

THE LOCAL STRUCTURE OF COMPACTIFIED JACOBIANS: DEFORMATION THEORY

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ABSTRACT. This paper studies the local geometry of compactified Jacobians constructed by Caporaso, Oda–Seshadri, Pandharipande, and Simpson. The main result is a presentation of the completed local ring of the compactified Jacobian of a nodal curve as an explicit ring of invariants described in terms of the dual graph of the curve. The authors have investigated the geometric and combinatorial properties of these rings in previous work, and consequences for compactified Jacobians are presented in this paper. Similar results are given for the local structure of the universal compactified Jacobian over the moduli space of stable curves at a point corresponding to an automorphism-free curve. In the course of the paper, we also review some of the literature on compactified Jacobians.

1. INTRODUCTION

This paper studies the local geometry of compactified Jacobians associated to nodal curves. These are projective varieties that play a role similar to that of the Jacobian variety for a non-singular curve. Recall that a Jacobian can be viewed as the moduli space of degree d line bundles on a non-singular curve, for a fixed integer d . A compactified Jacobian is an analogous parameter space associated to a nodal curve. A major barrier to constructing these spaces is that, while the moduli space of fixed degree line bundles on a nodal curve exists, it typically does not have nice properties: often it has infinitely many connected components (i.e. is not of finite type), and these components fail to be proper. To construct a well-behaved compactified Jacobian, one must modify the moduli problem. There are a number of different ways to do this, and the literature on the compactification problem is vast (e.g. [Ish78], [D’S79], [OS79], [Cap94], [Sim94], [Pan96], [Jar00], [Est01]).

Geometric Invariant Theory (GIT) provides a general framework for these types of compactification problems, and in this approach, the compactified Jacobian $\bar{J}^d(X)$ of a nodal curve X is constructed as a coarse moduli space of certain line bundles together with their degenerations: rank 1, torsion-free sheaves. The sheaves parameterized by $\bar{J}^d(X)$ are those rank 1, torsion-free sheaves that satisfy a certain numerical (semi-)stability condition. The specific condition differs from construction to construction. In the work of Simpson [Sim94], the condition is slope semi-stability with respect to an ample line bundle, while in the work of Oda–Seshadri [OS79], the condition is ϕ -semi-stability with respect to a linear algebraic parameter ϕ . We will refer to any of these numerical conditions as a semi-stability condition.

While $\bar{J}^d(X)$ is only a coarse moduli space, and typically many different semi-stable sheaves correspond to the same point of $\bar{J}^d(X)$, there is an open subset of $\bar{J}^d(X)$ parameterizing sheaves that satisfy a stronger numerical condition known as stability, and this open subset is a fine moduli

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space. In particular, isomorphism classes of stable sheaves are in bijection with points of the stable locus. A third condition, which plays an important role in this paper, is poly-stability. A semi-stable sheaf is said to be poly-stable if a Jordan-Hölder filtration of the sheaf, which is defined in terms of stability, splits. The main significance of this condition is that there is a natural bijection between the set of isomorphism classes of poly-stable sheaves and the points of $\bar{J}^d(X)$.

One motivation for constructing compactified Jacobians is that they provide degenerations of Jacobian varieties. Given a family of non-singular curves specializing to a nodal curve, the compactified Jacobian of the nodal fiber fits into a family that extends the family consisting of the Jacobians of the non-singular fibers. Note that because the coarse moduli space of stable curves \bar{M}_g does not admit a universal curve, this does not imply that the compactified Jacobians fit into a family over \bar{M}_g . However, Caporaso [Cap94] has constructed a family $\bar{J}_{d,g} \rightarrow \bar{M}_g$ of projective schemes that extends the Jacobian of the generic genus g curve. The elements of this family are not all compactified Jacobians in the sense just described, but they are closely related, and we call $\bar{J}_{d,g}$ the universal compactified Jacobian. We review this point in more detail in Fact 2.6 (3) and §2.8.

The main result of this paper describes the local geometry of both a compactified Jacobian $\bar{J}^d(X)$, and of the universal compactified Jacobian $\bar{J}_{d,g}$ at a point corresponding to an automorphism-free curve. In the statement of the theorem we use the term poly-stable; for a Simpson or Caporaso compactified Jacobian, this refers to slope poly-stability (§2, esp. §2.5). For an Oda-Seshadri compactified Jacobian, this refers to ϕ -poly-stability (§2.7 and Prop. 2.7).

Theorem A. *Let X be a nodal curve of arithmetic genus $g(X)$, let I be a rank 1, torsion-free sheaf on X , and let Σ be the set of nodes where I fails to be locally free. Set $\Gamma = \Gamma_X(\Sigma)$ to be the dual graph of any curve obtained from X by smoothing the nodes not in Σ . Fix an arbitrary orientation on Γ , and denote by $V(\Gamma)$, $E(\Gamma)$, and $s, t : E(\Gamma) \rightarrow V(\Gamma)$ the vertices, edges and source and target maps respectively. Set $b_1(\Gamma) = \#E(\Gamma) - \#V(\Gamma) + 1$. Let*

$$T_\Gamma := \prod_{v \in V(\Gamma)} \mathbb{G}_m, \quad \widehat{A(\Gamma)} := \frac{k[[X_e, Y_e : e \in E(\Gamma)]]}{(X_e Y_e : e \in E(\Gamma))} \quad \text{and} \quad \widehat{B(\Gamma)} := \frac{k[[X_e, Y_e, T_e : e \in E(\Gamma)]]}{(X_e Y_e - T_e : e \in E(\Gamma))}.$$

Define an action of the torus T_Γ on $\widehat{A(\Gamma)}$ and $\widehat{B(\Gamma)}$ by the rule that $\lambda = (\lambda_v)_{v \in V(\Gamma)} \in T_\Gamma$ acts as

$$\lambda \cdot X_e = \lambda_{s(e)} X_e \lambda_{t(e)}^{-1}, \quad \lambda \cdot Y_e = \lambda_{t(e)} Y_e \lambda_{s(e)}^{-1} \quad \text{and} \quad \lambda \cdot T_e = T_e.$$

Define complete local rings

$$R_I := \widehat{A(\Gamma)}[[W_1, \dots, W_{g(X)-b_1(\Gamma)}]] \quad \text{and} \quad R_{(X,I)} = \widehat{B(\Gamma)}[[W_1, \dots, W_{4g-3-b_1(\Gamma)-\#E(\Gamma)}]],$$

with actions of T_Γ induced by the actions on $\widehat{A(\Gamma)}$ and $\widehat{B(\Gamma)}$, and the trivial action on the remaining generators.

- (i) Suppose $\bar{J}^d(X)$ is a Caporaso–Pandharipande Jacobian associated to an automorphism-free curve, or an Oda–Seshadri Jacobian, or a Simpson Jacobian. If I is poly-stable and $x \in \bar{J}^d(X)$ is the point corresponding to I , there is an isomorphism

$$\widehat{\mathcal{O}}_{\bar{J}^d(X),x} \cong R_I^{T_\Gamma}$$

between the completed local ring of $\bar{J}^d(X)$ and the T_Γ -invariant subring of R_I .

- (ii) For the universal compactified Jacobian $\bar{J}_{d,g}$, if $\text{Aut}(X)$ is trivial, I is poly-stable and $y \in \bar{J}_{d,g}$ is the point corresponding to (X, I) , then there is an isomorphism

$$\widehat{\mathcal{O}}_{\bar{J}_{d,g},y} \cong R_{(X,I)}^{T_\Gamma}$$

between the completed local ring of $\bar{J}_{d,g}$ and the invariant subring of $R_{(X,I)}$.

Theorem A is a consequence of Theorems 5.10 and 6.1 (see also Remarks 5.9, 6.2). We discuss the proof in more detail below. The rings $\widehat{A(\Gamma)}$ appearing above are further studied in [CMKV]. In the notation of that paper, $\widehat{A(\Gamma)}$ is the completion of the ring $A(\Gamma)$ defined in [CMKV, Theorem A], and the action of T_Γ in both papers is the same. It is shown in [CMKV, Theorem A] that the invariant subring $A(\Gamma)^{T_\Gamma}$ is isomorphic to the cographic ring $R(\Gamma)$ defined in [CMKV, Definition 1.4]. In particular, the completed local ring of the compactified Jacobian is isomorphic to a power series ring over a completion of the cographic toric face ring $R(\Gamma)$. A number of geometric properties of cographic rings are established in [CMKV], and some consequences for compactified Jacobians are discussed in Theorem B below. We intend to study the geometric and combinatorial properties of the rings $\widehat{B(\Gamma)}^{T_\Gamma}$ in more detail in future work.

Theorem B. *Suppose $\bar{J}^d(X)$ is a Caporaso–Pandharipande compactified Jacobian associated to an automorphism-free curve, or an Oda–Seshadri Jacobian, or a Simpson Jacobian. Then:*

- (i) *$\bar{J}^d(X)$ has Gorenstein, semi log-canonical (slc) singularities. In particular, $\bar{J}^d(X)$ is semi-normal.*
- (ii) *Let I be a poly-stable rank 1, torsion-free sheaf that corresponds to a point $x \in \bar{J}^d(X)$. Then x lies in the smooth locus of $\bar{J}^d(X)$ if and only if I fails to be locally free only at separating nodes of the dual graph of X .*

The proof is given at the end of §6. In [CMKV] it is shown that a number of further properties of cographic rings can be determined from elementary combinatorics of the graph $\Gamma = \Gamma_X(\Sigma)$ introduced in Theorem A. For instance, that paper provides combinatorial formulas giving the embedding dimension and the multiplicity of $\widehat{\mathcal{O}}_{\bar{J}^d(X),x}$, as well as a description of the irreducible components and the normalization of this ring. The reader is directed to §7 and [CMKV] for more details. We also point out that it is well-known that the completed local ring of $\bar{J}^d(X)$ at a *stable* point is isomorphic to a completed product of nodes. Using Theorem A and the results of [CMKV] one can construct examples of compactified Jacobians whose structure at a strictly semi-stable point is more complicated (see esp. § 7.2).

The approach we take to proving the theorems is via deformation theory. The basic strategy is to show that the local structure of a compactified Jacobian is given by the quotient of a mini-versal space by the action of an automorphism group. We sketch the proof for the Simpson Jacobian $\bar{J}_{L,d}$ (the other cases are handled similarly). To begin, the miniversal deformation ring R_I of a rank 1, torsion-free sheaf I on a nodal curve X is well understood, and from this description, it is easy to define a natural action $\text{Aut}(I)$ on R_I (in fact, this is the ring R_I and group action of T_Γ described in Theorem A; see Corollary 3.17 and Remark 5.9). We then show that when I is poly-stable, the ring of invariants $R_I^{\text{Aut}(I)}$ is isomorphic to the local ring of the moduli space at the point corresponding to I , completing the proof.

The main difficulty is establishing this last point. Our approach is to consider the GIT constructions of the spaces, employ the Luna Slice Theorem, and then use a theorem of Rim on reductive group actions on mini-versal rings. More precisely, we start with the fact that the compactified Jacobian is constructed as a GIT quotient of a suitable Quot scheme $\text{Quot}(\mathcal{O}_X^r)$ by the action of a reductive group G (see §2 for a review of the constructions). We check that the complete local ring R_x of a Luna Slice at a poly-stable point $x \in \text{Quot}(\mathcal{O}_X^r)$ is a miniversal deformation ring for the corresponding sheaf I (Lemma 6.4). Thus R_x is (non-canonically) isomorphic to R_I . The GIT set-up provides an action of the stabilizer G_x of x on the ring R_x , and the Luna Slice Theorem states that the invariant ring $R_x^{G_x}$ gives the local structure of the GIT quotient. Finally we use a theorem of Rim (Fact 5.4) to compare the action of G_x on R_x to our chosen action of $\text{Aut}(I)$ on R_I , and establish that the invariant rings associated to the two groups are isomorphic, completing the proof (Thms. 5.10, 6.1).

The main results of this paper suggest several questions for further study. First, the authors hope to use the results of this paper to study the singularities of the theta divisor of a stable curve. The theta divisor is an ample divisor on a compactified Jacobian. When the theta divisor is associated to a non-singular curve, there is a large body of work describing the singularities, and the authors hope to extend that work. In a different direction, the authors to apply Theorem A to further study the singularities of $\bar{J}_{d,g}$. In particular, they hope to answer the following question: does $\bar{J}_{d,g}$ have canonical singularities? In [BFV, Thm. 1.4], it is shown that the answer is “yes” when $\bar{J}_{d,g}$ has quotient singularities, which happens precisely when $\gcd(d+1-g, 2g-2) = 1$. Theorem 1.2 of [BFV] computes the Kodaira dimension of $\bar{J}_{d,g}$ under the assumption that $\gcd(d+1-g, 2g-2) = 1$, and a positive answer to the question about canonical singularities would allow this hypothesis to be removed.

To answer that question, it is necessary to extend Item(ii) of Theorem A to the case where X admits non-trivial automorphisms. If one assumes that X does not admit an automorphism of order p (the characteristic of the group field), then the completed local ring is isomorphic to an appropriate invariant subring (Thm. 6.1), but we do not compute that ring here. Extending Theorem A to the case where X admits an automorphism of order p is more challenging because then $\text{Aut}(X, I)$ is reductive, but not linearly reductive. Linear reductivity is crucial in two places: in the proof of Theorem 6.1, which uses the Luna Slice Theorem, and Theorem 5.10, which uses a result of Rim. The technical obstacles to weakening the linear reductivity hypotheses are discussed after the proofs of the two theorems.

Positive characteristic issues also appear in Fact 2.6, which relates the fibers of $\bar{J}_{d,g} \rightarrow \bar{M}_g$ to compactified Jacobians. That result is only stated in characteristic 0, and it would be interesting to know if the result remains valid in positive characteristic. This is discussed in greater detail immediately after the proof of the fact.

There are approaches to describing the local structure of a compactified Jacobian different from the approach taken here. Alexeev has proven that compactified Jacobians are stable semi-abelic varieties ([Ale04, Thm. 5.1]), and consequently can be described by Mumford’s construction. In Mumford’s work, one compactifies a semi-abelian variety by first forming the projectivization of a (non-finitely generated) graded algebra and then quotienting out by a lattice. This procedure provides direct access to the local structure of the compactification, and thus Alexeev’s work provides another approach to studying the local structure of compactified Jacobians. It would be very interesting to compare the descriptions arising from this approach to the descriptions given in this paper, but we do not pursue this topic here.

The results of Theorem B are related to some results in the literature. Specifically, it was known that $\bar{J}^d(X)$ is seminormal [Ale04, Thm. 5.1] and Gorenstein [AN99, Lemma 4.1]. In personal correspondence, Alexeev has explained to the authors that the techniques of those papers can also be used to establish the fact that $\bar{J}^d(X)$ is semi-log canonical. The description of the smooth locus of $\bar{J}^d(X)$ is certainly well-known to the experts (see e.g. [Cap94, Thm. 6.1(3)], [Cap08, Thm. 7.9(iii)], [Cap09, Fact 4.1.5(iv)], [MV, Fact 1.19(ii)]); however, it seems that a proof has not appeared in print.

Finally, one motivation for the approach we take in this paper is for studying the singularities of the theta divisor of a compactified Jacobian. Theta divisors can be viewed as moduli spaces of rank 1, torsion-free sheaves admitting non-trivial sections, and from this perspective we find it advantageous to have a description of the local structure of the compactified Jacobian in terms of deformations. We will investigate singularities of theta divisors in future work.

This paper is organized as follows. We review the literature on compactified Jacobians in §2 with the goal of collecting the facts needed to prove Theorems A and B. The proofs of the main theorems begin in §3, where we develop the deformation theory needed to compute deformation rings parameterizing deformations of a rank 1, torsion-free sheaf. These rings admit natural actions

of automorphism groups, which are described in the next two sections. The structure of the automorphism groups is studied in §4, and then those results are used in §5 to compute group actions. Finally, in §6 we prove the main results of this paper by using the Luna Slice Theorem to relate the local structure of a compactified Jacobian to a deformation ring. In §7 we describe some examples using results of [CMKV].

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Conventions.

- 1.1. k will denote an algebraically closed field (of arbitrary characteristic).
- 1.2. All **schemes** are k -schemes, and all morphisms are implicitly assumed to respect the k -structure.
- 1.3. A **curve** is a connected, complete, reduced scheme (over k) of pure dimension 1.
- 1.4. A **subcurve** Y of a curve X is a proper, closed k -scheme $Y \hookrightarrow X$ that is reduced and of pure dimension 1 (but possibly disconnected).
- 1.5. A **family of curves** is a proper, flat morphism $X \rightarrow T$ whose geometric fibers are curves.
- 1.6. The **genus** $g(X)$ of a curve X is $g(X) := h^1(X, \mathcal{O}_X)$.
- 1.7. A **family of coherent sheaves** on a family of curves $X \rightarrow T$ is a \mathcal{O}_T -flat, finitely presented \mathcal{O}_X -module I .
- 1.8. A coherent sheaf I on a nodal curve X is said to be of **rank 1** if I has rank 1 at every generic point.
- 1.9. A coherent sheaf I on a nodal curve X is said to be **pure** if for every non-zero subsheaf $J \subseteq I$ the dimension of the support of J is equal to the dimension of the support of I .
- 1.10. A coherent sheaf I on a nodal curve X is said to be **torsion-free** if it is pure and the support of I is X .

2. PRELIMINARIES ON COMPACTIFIED JACOBIANS

Here we review some facts about compactified Jacobians with the goal of collecting the results needed to prove Theorem 6.1. These compactified Jacobians are coarse moduli spaces of sheaves constructed using Geometric Invariant Theory (GIT). This section is organized as follows. We begin by reviewing the language of co-representability, which provides a convenient language for discussing coarse moduli spaces. We then review several of the GIT constructions from the literature. Finally, we describe the relations between the different constructions.

2.1. Representable and co-representable functors. Recall that the **functor of points** of a k -scheme F is the set-valued functor defined by the rule

$$(2.1) \quad T \mapsto \mathrm{Hom}_k(T, F).$$

Yoneda's Lemma states that a k -scheme can be recovered from its functor of points, and so we will abuse notation and write $F: k\text{-Sch.} \rightarrow \text{Sets}$ for the functor of points of the scheme F . We say that a functor is **representable** if it is isomorphic to the functor of points of some k -scheme. If $F^\sharp: k\text{-Sch.} \rightarrow \text{Sets}$ is a functor and (F, π) is a pair consisting of a k -scheme F and an isomorphism $\pi: F^\sharp \xrightarrow{\sim} F$, then we say that (F, π) **represents** F^\sharp .

Many of the functors we consider here are not representable, but satisfy the weaker condition of co-representability. Given a functor $F^\sharp: k\text{-Sch.} \rightarrow \text{Sets}$ and a natural transformation $\pi: F^\sharp \rightarrow F$ from F^\sharp to a k -scheme F , we say that (F, π) **co-represents** F^\sharp if π is universal with respect to natural transformations from F^\sharp to a k -scheme. In other words, if $t: F^\sharp \rightarrow T$ is any natural transformation to a k -scheme T , then t factors uniquely as $t = \hat{t} \circ \pi$ for some morphism $\hat{t}: F \rightarrow T$. In the definition of co-representability, some authors require that π induces a bijection $F^\sharp(k) \rightarrow F(k)$ on k -valued points, but we do not require this. If (F, π) co-represents F^\sharp and this property persists under flat (resp. arbitrary) base change $T \rightarrow F$, then we say that (F, π) **uniformly** (resp. **universally**) **co-represents** F^\sharp .

Observe that a given k -scheme F may co-represent two non-isomorphic functors. In fact, there is a class of natural transformations with the property that a scheme co-represents the target if and only if it co-represents the source; namely, the (étale) local isomorphisms. Suppose $G^\sharp \rightarrow F^\sharp$ is a natural transformation. We say that $G^\sharp \rightarrow F^\sharp$ is a (étale) **local surjection** if for all $T \in k\text{-Sch}$ and $y \in F^\sharp(T)$ there exists an étale cover $\{T_i \rightarrow T\}$ of T and elements $\{x_i \in G^\sharp(T_i)\}$ with x_i mapping to the restriction $y|_{T_i} \in F^\sharp(T_i)$. Similarly, we say that the given natural transformation is a (étale) **local injection** if for all $T \in k\text{-Sch}$ and $x, x' \in G^\sharp(T)$ mapping to the same element of $F^\sharp(T)$ there exists an étale cover $\{T_i \rightarrow T\}$ such that the restrictions of x and x' to T_i coincide in $G^\sharp(T_i)$. A natural transformation that is both a local surjection and a local injection is a (étale) **local isomorphism**. The reader familiar with the étale topology will recognize that a local isomorphism is a natural transformation that induces an isomorphism on the étale sheaves associated to F^\sharp and G^\sharp .

If $G^\sharp \rightarrow F^\sharp$ is a local isomorphism, then the co-representability properties of G^\sharp are equivalent to those of F^\sharp . More precisely, if $G^\sharp \rightarrow F^\sharp$ is a local isomorphism and (F, π) co-represents F^\sharp , then the composition $G^\sharp \rightarrow F^\sharp \xrightarrow{\pi} F$ co-represents G^\sharp . Furthermore, if (G, π) co-represents G^\sharp , then π factors uniquely as $G^\sharp \rightarrow F^\sharp \xrightarrow{\bar{\pi}} G$, where $\bar{\pi}$ co-represents F^\sharp . These results are consequences of the fact that a representable functor is a sheaf with respect to the étale topology.

For example, consider the statement that if (F, π) co-represents F^\sharp , then the composition $G^\sharp \rightarrow F^\sharp \xrightarrow{\pi} F$ co-represents G^\sharp . To prove this, it is enough to show that any morphism $G^\sharp \rightarrow T$ to a k -scheme T uniquely factors as $G^\sharp \rightarrow F^\sharp \rightarrow T$. In fact, by Yoneda's Lemma, it is enough to factor $G^\sharp(S) \rightarrow T(S)$ for all k -schemes S . Given S and $y \in F^\sharp(S)$, there is an étale cover $\{S_i \rightarrow S\}$ and elements $x_i \in G^\sharp(S_i)$ lifting the restrictions of y . Let t_i denote the image of x_i in $T(S_i)$. Because T is an étale sheaf, there is a unique element $t \in T(S)$ lifting the elements t_i . One may verify that the rule $y \mapsto t$ defines a map $F^\sharp(S) \rightarrow T(S)$ factoring $F^\sharp(S) \rightarrow T(S)$, which establishes the claim. The statement about (G, π) can be proven by a similar argument.

The language of co-representability is used to define coarse moduli spaces. As an illustration, let's recall the description of the coarse moduli space of stable curves using this language. Define a functor $\overline{M}_g^\sharp: k\text{-Sch} \rightarrow \text{Sets}$ by setting $\overline{M}_g^\sharp(T)$ equal to the set of isomorphism classes of families of genus g stable curves over T . The coarse moduli space of stable curves \overline{M}_g is the k -scheme that co-represents \overline{M}_g^\sharp . Inside of \overline{M}_g^\sharp , one can consider the subfunctor $(\overline{M}_g^\circ)^\sharp$ parameterizing automorphism-free curves, and this subfunctor is representable by an open subscheme \overline{M}_g° of \overline{M}_g .

2.2. An overview of compactified Jacobian functors. We now give an overview of the compactified Jacobians studied in this paper. Here we briefly describe different approaches to their constructions. We defer giving detailed definitions until later in the section, after we have reviewed some basic facts from GIT.

In this paper, we primarily study compactified Jacobians constructed as moduli spaces of rank 1, torsion-free sheaves satisfying a numerical condition, or semi-stability condition, on the multi-degree. One common numerical condition to impose is *slope* semi-stability. If (X, L) is a pair

consisting of a nodal curve X with ample line bundle L , then the semi-stability condition on a coherent sheaf I is a positivity condition on the Hilbert polynomial of I , taken with respect to L . (See Sect. 2.6 for the precise definition.) There is a large body of work on moduli spaces of slope semi-stable sheaves on polarized schemes, and we will primarily make use of the work of Simpson ([Sim94]). Simpson works over the complex numbers \mathbb{C} , and the relevant result from his paper is that the moduli functor $\bar{J}_{L,d}^\sharp(X)$ parameterizing slope semi-stable rank 1, torsion-free sheaves of degree d on X is universally co-represented by a projective scheme $\bar{J}_{L,d}(X)$. Following tradition, we call $\bar{J}_{L,d}^\sharp(X)$ the **Simpson Jacobian functor** of degree d and $\bar{J}_{L,d}(X)$ the **Simpson Jacobian**.

There are several ways to generalize the construction of $\bar{J}_{L,d}(X)$. First, in Simpson's work, the slope semi-stability condition arises as GIT semi-stability with respect to a natural linearization, and alternative compactifications can be constructed by varying the linearization. Some such compactifications were constructed by Oda and Seshadri in [OS79], where the authors constructed what are called ϕ -compactified Jacobians $\bar{J}_\phi(X)$. Like the Simpson Jacobian, $\bar{J}_\phi(X)$ parameterizes sheaves satisfying a semi-stability condition, but the condition is not slope semi-stability, but rather ϕ -semi-stability, which is a numerical condition defined in terms of an auxiliary linear algebraic parameter ϕ (whose definition is recalled in Sect. 2.7). The basic existence theorem is similar to the theorem concerning the existence of $\bar{J}_{L,d}(X)$: the natural functor $\bar{J}_\phi(X)$ parameterizing ϕ -semi-stable rank 1, torsion-free sheaves, which we call the **ϕ -compactified Jacobian functor**, is co-represented by a projective scheme called the **ϕ -compactified Jacobian**.

In a different direction, one could try to construct an analogue of $\bar{J}_{L,d}(X)$ in a relative setting. In defining the Simpson Jacobian, we began with a polarized nodal curve (X, L) , but one could instead start with a *family* X/S of nodal curves (over some base S) and an S -relatively ample line bundle L . Simpson's work applies in this more general context, and he shows that the **relative Simpson Jacobian functor** $\bar{J}_{L,d}^\sharp(X/S)$ of degree d , defined in the expected manner, is universally co-represented by an S -projective scheme $\bar{J}_{L,d}(X/S)$ called the **relative Simpson Jacobian**, provided S is finitely generated over \mathbb{C} .

Simpson's work does not provide a family over \bar{M}_g , as \bar{M}_g does not admit a universal family of curves. Despite this, Caporaso and Pandharipande have constructed a family $\bar{J}_{d,g} \rightarrow \bar{M}_g$ that plays the role of the relative compactified Jacobian. Given integers d and g with $g \geq 2$, the **universal compactified Jacobian functor** $\bar{J}_{d,g}^\sharp: k\text{-Sch.} \rightarrow \text{Sets}$ is defined to be the functor parameterizing slope semi-stable, rank 1, torsion-free sheaves of degree d on genus g stable curves up to equivalence (see Sect. 2.8). Here the stability condition is taken with respect to the relative dualizing sheaf. The basic existence theorem asserts that $\bar{J}_{d,g}^\sharp$ is co-representable by a projective scheme $\bar{J}_{d,g}$ called the **universal compactified Jacobian**. Furthermore, this scheme admits a morphism to \bar{M}_g that lifts the forgetful morphism $\bar{J}_{d,g}^\sharp \rightarrow \bar{M}_g^\sharp$.

This description of $\bar{J}_{d,g}$ is not in Caporaso's original work, but is in rather the later work of Pandharipande. In [Cap94], Caporaso constructs $\bar{J}_{d,g}$ as a coarse moduli space parameterizing line bundles on (possibly unstable) nodal curves. We omit a detailed discussion of the construction, because Pandharipande's description of $\bar{J}_{d,g}$ most naturally relates to the other compactified Jacobians we discuss here. We direct the interested reader to ([Cap94], [Pan96, §10]).

There are several approaches to constructing compactified Jacobians that do not use Geometric Invariant Theory. One approach is to construct compactifications using the theory of degenerations of abelian varieties (as in [FC90]). Such a program has been worked out by many mathematicians, including Alexeev ([Ale02]), Namikawa ([Nam76]), and Nakamura ([Nak75]). The reader interested in learning more about these constructions and their relation to the compactified Jacobians studied here is directed to [Ale04], which discusses the relation between Alexeev's semi-abelic varieties and various coarse moduli spaces of sheaves.

We also avoid any explicit discussion of algebraic stacks in this paper, but stack-theoretic techniques provide a third approach to the compactification problem. Closest to the compactified Jacobians studied in this paper are the stacks appearing in [Mel09]. There Melo constructed an Artin stack whose coarse space is $\bar{J}_{d,g}$. A construction similar to Caporaso's appears in [FTT], where the authors construct what they call the stack of Gieseker bundles. Like Caporaso's space, their moduli space parameterizes certain distinguished pairs consisting of a nodal curve and a line bundle, but they construct an Artin stack rather than a scheme. Another approach to extending the Jacobian of the general genus g curve can be found in the work of Jarvis ([Jar00]). He constructs an Artin stack parameterizing what are called roots of line bundles. It would be interesting to know if the ideas of this paper can be used to describe the local geometry of these stacks, but we do not pursue the topic here.

2.3. Geometric Invariant Theory. We review some background from Geometric Invariant Theory (GIT) that is needed later. The coarse moduli spaces we study here are all constructed using GIT, and in proving statements about them, we will need to make use of their construction, and not just the fact of their existence.

Recall that GIT is a tool for constructing a quotient of an algebraic k -scheme Q by the action of a reductive group G . Given an auxiliary ample line bundle $\mathcal{O}(1)$ together with a lift of the action of G on Q to an action on $\mathcal{O}(1)$ (i.e. a linearization), there is distinguished open subscheme Q^{ss} of Q that consists of points that are semi-stable with respect to the linearized action. The significance of Q^{ss} is that it admits a **uniform categorical quotient** that we define to be the GIT quotient of Q , written $Q//G$. That is, there exists a pair $(Q^{\text{ss}}/G, \pi)$ consisting of an algebraic scheme Q^{ss}/G and a G -invariant map

$$\pi: Q^{\text{ss}} \rightarrow Q^{\text{ss}}/G$$

with the property that

- (1) π is universal among all G -invariant maps out of Q^{ss} (i.e. $(Q^{\text{ss}}/G, \pi)$ is a **categorical quotient**);
- (2) this property persists after base change by an arbitrary flat morphism $T \rightarrow Q^{\text{ss}}/G$ (i.e. the categorical quotient is **uniform**).

The quotient is always a quasi-projective scheme, and has the further property that the quotient map π is affine and universally submersive (i.e. a subset $U \subseteq Q^{\text{ss}}/G$ is open if and only if $\pi^{-1}(U)$ is open in Q^{ss} , and this property persists under arbitrary base change $T \rightarrow Q^{\text{ss}}/G$). When the characteristic of k is 0, the pair (Q^{ss}, π) is actually a **universal categorical quotient**, so Property (1) persists under base change by an arbitrary morphism $T \rightarrow Q^{\text{ss}}/G$ (rather than just by a flat morphism).

The local structure of $Q//G$ is described by the Luna Slice Theorem, which compares $Q//G$ to the quotient of a certain model G -space. For the remainder of §2.3, we assume that Q is affine, so $Q = Q^{\text{ss}}$ and $Q//G$ is the categorical quotient ([MFK94, Thm. 1, p. 27]). The model scheme is $G \times_H V$, whose definition we now review. Suppose $H \subset G$ is a reductive subgroup and V a scheme with a left H -action. The product $G \times V$ carries an H -action defined by

$$h \cdot (g, x) := (gh^{-1}, h \cdot x),$$

and we write $G \times_H V$ for the categorical quotient. This quotient admits a left action of G defined by the translation action on the first factor.

The two projections out of $G \times V$ induce morphisms

$$\begin{aligned} p: G \times_H V &\rightarrow V/H, \\ q: G \times_H V &\rightarrow G/H. \end{aligned}$$

The first map is G -invariant and realizes V/H as the quotient of $G \times_H V$ by G . The map q is equivariant and can often be described as a contraction onto an orbit. To be precise, suppose we are given an element $v_0 \in V$ fixed by H . One may verify that the image of $(e, v_0) \in G \times V$ in $G \times_H V$ has stabilizer H , and the associated orbit map defines a section of q .

The Luna Slice Theorem provides sufficient conditions for $Q//G$ to be étale locally isomorphic to V/H for a suitable H and V . More precisely, let x be a point of Q with stabilizer H . Given any affine, locally closed subscheme $V \subset Q$ that contains x and is stabilized by H (i.e. $H \cdot V \subset V$), the action map induces a G -equivariant morphism

$$G \times_H V \rightarrow Q.$$

We say that V is a **slice** at x if the following conditions are satisfied:

- (1) the morphism $G \times_H V \rightarrow Q$ is étale;
- (2) the image of $G \times_H V \rightarrow Q$ is an open affine $U \subset Q$ that is π -saturated (i.e. for each $u \in U$, $\pi^{-1}(\pi(u)) \subseteq U$);
- (3) the induced morphism $(G \times_H V)/G \rightarrow U/G$ is étale;
- (4) the induced morphism $G \times_H V \rightarrow U \times_{U/G} V/H$ is an isomorphism.

Note in particular that condition (3) together with the observation above on the map p implies that there is an étale morphism

$$V/H \xrightarrow{\text{ét}} Q//G.$$

The original Luna Slice Theorem [Lun73, pg. 97] states that in characteristic zero a slice exists provided that x is (GIT-) **poly-stable**, i.e. the orbit of x is closed. When x has a closed orbit, Matsushima's criterion implies that the stabilizer H is reductive ([Mat60] for $k = \mathbb{C}$; [Ric77] for k arbitrary). Bardsley and Richardson have extended the Luna Slice Theorem to arbitrary characteristic. With no assumptions on $\text{char}(k)$, they prove that a slice exists provided the orbit of x is closed and the stabilizer H is reduced and linearly reductive ([BR85, Prop. 7.6]; the condition in loc. cit. that the orbit is separable is equivalent to our condition that H is reduced). We will use the Luna Slice Theorem to study a compactified Jacobian by applying the theorem to a Quot scheme, and we now review these objects.

2.4. Quot Schemes. Quot schemes were first introduced in [Gro95], and a recent review of the literature can be found in [Nit05]. Let $(X/S, \mathcal{O}_X(1))$ be a pair consisting of a projective morphism $X \rightarrow S$ and a relatively very ample line bundle $\mathcal{O}_X(1)$ on X . Given a locally finitely presented \mathcal{O}_X -module E and a polynomial $P(t) \in \mathbb{Q}[t]$, the associated **relative Quot functor** $\text{Quot}^\sharp(E, P(t))$ is defined by setting $\text{Quot}^\sharp(E, P(t))(T)$, for an S -scheme T , equal to the set of isomorphism classes of locally finitely presented, T -flat quotients $E_T \twoheadrightarrow F$ on $X_T := T \times_S X$ with the property that on each fiber the Hilbert polynomial of F (with respect to $\mathcal{O}_X(1)$) is $P(t)$; here E_T is the pull-back of E . Grothendieck's theorem is that this functor is representable by a projective S -scheme $\text{Quot}(E, P(t))$ called the **relative Quot scheme**. Here we will only make use of the case where $E = \mathcal{O}_X(b)^{\oplus r}$, for some r and b .

The scheme $\text{Quot}(\mathcal{O}(b)_X^{\oplus r}, P(t))$ carries a natural action of the group SL_r given by changing coordinates on $\mathcal{O}_X^{\oplus r}$, and this action can be linearized using the construction of the Quot scheme. Indeed, the construction realizes $\text{Quot}(\mathcal{O}(b)_X^{\oplus r}, P(t))$ as a closed subscheme of a Grassmannian, which in turn embeds into a projective space via the Plücker coordinates. The resulting ample line bundle admits a lift of the action of SL_r , providing the linearization.

2.5. Slope stability. We recall some facts about slope stability. Given a polarized nodal curve (X, L) and a coherent sheaf I on X , the **slope** $\mu_L(I)$ of I with respect to L is defined to be a/r , where a and r are coefficients of the Hilbert polynomial $P_L(I, t) := r \cdot t + a$ of I with respect to L . The sheaf I is said to be **slope semi-stable** if it is pure and satisfies $\mu_L(I) \leq \mu_L(J)$ for all pure

quotients $I \twoheadrightarrow J$ with 1-dimensional support $\text{Supp}(J)$. If this inequality is always strict, then we say that I is **slope stable**. The sheaf I is said to be **slope poly-stable** if it is slope semi-stable and isomorphic to a direct sum of slope stable sheaves. (This condition can be restated in terms of the splitting of a Jordan-Hölder filtration)

There are some obvious pure quotients of a rank 1, torsion free-sheaf I . Let $i: Y \hookrightarrow X$ be the inclusion of a subcurve. The sheaf $i^*(I)$ may contain torsion, but its torsion-free quotient, which we denote by I_Y , is pure. Adjunction provides a natural surjection $I \twoheadrightarrow i_*(I_Y)$, so if I is slope semi-stable, then we must have $\mu_L(I) \leq \mu_L(i_*(I_Y))$. When I is a rank 1, torsion-free sheaf on a nodal curve, these inequalities actually imply semi-stability:

Fact 2.1. *Let (X, L) be a pair consisting of a nodal curve X and an ample line bundle L . If I is a rank 1, torsion-free sheaf on X , then I is slope semi-stable (with respect to L) if and only if*

$$(2.2) \quad \mu_L(I) \leq \mu_{i^*L}(I_Y)$$

for all subcurves $i: Y \hookrightarrow X$.

This fact is the content of the lemma below.

Lemma 2.2. *Let I be a rank 1, torsion-free sheaf on a nodal curve X . If $q: I \twoheadrightarrow J$ is a quotient map whose quotient J is a pure sheaf with 1-dimensional support $Y := \text{Supp}(J)$, then q factors as*

$$I \rightarrow i_*(I_Y) \xrightarrow{\bar{q}} J,$$

where $\bar{q}: i_*(I_Y) \rightarrow J$ is an isomorphism.

Proof. First, we show that the ideal of $Y \subset X$ acts trivially on J . In other words, given an affine open subset $U \subset X$, a regular function $f \in H^0(U, \mathcal{O}_X)$ vanishing on Y , and a local section $s \in H^0(U, J)$ of J , we must show $f \cdot s = 0$. We do so by studying the support of $f \cdot s$.

This support is certainly contained in $Y \cap U$, and in fact, it cannot contain any irreducible component of $Y \cap U$. Indeed, if $y \in Y \cap U$ is a generic point, then by assumption, f has zero image in the fiber $k(y) = \mathcal{O}_{X,y}/\mathfrak{m}_y$, which coincides with $\mathcal{O}_{X,y}$. Thus, $y \notin \text{Supp}(f \cdot s)$, or in other words, $\text{Supp}(f \cdot s)$ is 0-dimensional. This forces $f \cdot s = 0$, as J is pure.

The lemma now follows. As the ideal of Y acts trivially on J , we may consider this sheaf as a coherent sheaf J_Y on Y satisfying $i_*(J_Y) = J$. By adjunction, q corresponds to a map $i^*(I) \rightarrow J_Y$. The torsion submodule of $i^*(I)$ must be contained in the kernel of this map as J_Y is pure, so there is an induced map $I_Y \rightarrow J_Y$. Forming the direct image under i_* , we obtain a map $\bar{q}: i_*(I_Y) \rightarrow i_*(J_Y) = J$ factoring q .

To complete the proof, we must show that \bar{q} is an isomorphism. By construction, this map is surjective, so the only issue is injectivity. At every generic point, the map induced by \bar{q} is a surjection from a free, rank 1 module to a non-zero module, hence an isomorphism. The support of $\ker(\bar{q})$ is thus at most 0-dimensional, hence $\ker(\bar{q}) = 0$ as $i_*(I_Y)$ is pure. This completes the proof. \blacksquare

2.6. The Simpson Construction. In [Sim94], Simpson constructs a coarse moduli space of semi-stable sheaves as the GIT quotient of a Quot scheme with respect to the linearized group action just described. Simpson works with a projective scheme $(X, \mathcal{O}(1))$ (or, more generally, a projective family of schemes), and shows that that GIT quotient is a coarse moduli space of sheaves satisfying the condition of p -semi-stability. In general, p -semi-stability is a condition on the Hilbert polynomial of a coherent sheaf (with respect to $\mathcal{O}(1)$), but here we are only concerned with the case that X is a nodal curve, in which case p -semi-stability coincides with slope semi-stability. We define the Simpson compactified Jacobian of a nodal curve X to be a connected component of the associated Simpson moduli space of slope semi-stable sheaves. We now give a more detailed description.

Simpson works with a pair $(X/S, L)$ consisting of a finitely generated \mathbb{C} -scheme S , a projective S -scheme X , and a relatively ample line bundle L (Langer ([Lan04]) and Maruyama ([Mar96]) have extended Simpson's work to the case of a more general base S , but we will not use this). Given a polynomial $P(t) \in \mathbb{Q}[t]$, we define the **Simpson moduli functor** $M^\sharp(\mathcal{O}_X, P)$ to be the functor whose T -valued points are families of coherent sheaves that are fiber-wise slope semi-stable with Hilbert polynomial P (with respect to L). Simpson [Sim94, Thm. 1.21] has proven that there exists a projective S -scheme $M(\mathcal{O}_X, P) \rightarrow S$, the **Simpson moduli space**, that universally co-represents $M^\sharp(\mathcal{O}_X, P)$.

Now suppose we specialize to the case where $X \rightarrow S$ is a projective family of nodal curves of genus g . Assume the fiber-wise degree of L is constant and equal to $\deg(L_s)$, and define the Hilbert polynomial

$$(2.3) \quad P_d(t) := \deg(L_s) \cdot t + d + 1 - g.$$

The polynomial P_d is the Hilbert polynomial of a degree d line bundle, so every slope semi-stable rank 1, torsion-free sheaf of degree d corresponds to a point of $M(\mathcal{O}_X, P_d)$. However, there may be points of this scheme that do not correspond to rank 1, torsion-free sheaves (see [LM05, Ex. 2.2]), so we do not define $M(\mathcal{O}_X, P_d)$ to be the relative Simpson Jacobian. Instead, we first define the **relative Simpson Jacobian functor** of degree d to be the functor

$$\bar{J}_{L,d}^\sharp(X/S): S\text{-Sch.} \rightarrow \text{Sets}$$

whose T -valued points are families of coherent sheaves that are fiber-wise slope semi-stable, rank 1, torsion-free, and of degree d . Below we prove that $\bar{J}_{L,d}^\sharp(X/S)$ is co-represented by a projective S -scheme $\bar{J}_{L,d}(X/S) \rightarrow S$ that we call the **relative Simpson Jacobian** of degree d . When $S = \text{Spec}(\mathbb{C})$, we drop the prefix “relative” and write $\bar{J}_{L,d}(X)$ in place of $\bar{J}_{L,d}(X/\text{Spec}(\mathbb{C}))$.

$\bar{J}_{L,d}^\sharp(X/S)$ is a subfunctor of $M^\sharp(\mathcal{O}_X, P_d)$, and the existence of $\bar{J}_{L,d}(X/S)$ can easily be deduced from Simpson's work, but we need to use the construction of $M(\mathcal{O}_X, P_d)$, not just the fact of its existence, so we first review the relevant facts. Temporarily write $\mathcal{O}_X(1)$ for L . Given b sufficiently large, Simpson sets $r := P_d(b)$ and considers $\text{Quot}(\mathcal{O}_X(-b)^{\oplus r}, P_d)$. There is a closed and open subscheme ([Sim94, pg. 66]) $Q^\circ \subseteq \text{Quot}(\mathcal{O}_X(-b)^{\oplus r}, P_d)$ that parameterizes quotient maps

$$q: \mathcal{O}_X(-b)^{\oplus r} \twoheadrightarrow F$$

satisfying the following additional conditions:

- (1) $H^1(X, F(b)) = 0$;
- (2) $q \otimes 1: H^0(X, \mathcal{O}_X^{\oplus r}) \rightarrow H^0(X, F(b))$ is an isomorphism;
- (3) F is a semi-stable sheaf.

The natural action of SL_r restricts to an action on Q° , and by definition $M(\mathcal{O}_X, P_d) = Q^\circ // \text{SL}_r$. We now use this description of $M(\mathcal{O}_X, P_d)$ to prove that $\bar{J}_{L,d}(X/S)$ is universally co-representable.

Simpson proves this theorem by relating GIT stability to slope stability. Specifically, Simpson proves that a point of the Quot scheme corresponding to $q: \mathcal{O}(-b)^{\oplus r} \twoheadrightarrow I$ is GIT (resp. semi-, poly-)stable if and only if I is (resp. semi-, poly-)stable with respect to the polarization L . GIT stability is related to slope stability in [Sim94, Cor. 1.20], while the semi-stability and poly-stability conditions are related in [Sim94, Thm. 1.19] and [Sim94, Pf. of Thm. 1.21] respectively.

Lemma 2.3. *Let $(X/S, L)$ be a pair consisting of a family of genus g , nodal curves with relatively ample line bundle L . Then the subset of $\bar{J}_{L,d}(X/S) \subset M(\mathcal{O}_X, P_d)$ corresponding to rank 1, torsion-free sheaves is closed-and-open and universally co-represents the relative Simpson compactified Jacobian functor $\bar{J}_{L,d}^\sharp(X/S)$.*

Proof. First, a remark about the definition of $\bar{J}_{L,d}(X/S)$. We stated that a given point $x \in M(\mathcal{O}_X, P_d)$ lies in $\bar{J}_{L,d}(X/S)$ iff it corresponds to a rank 1, torsion-free sheaf, but x does not correspond to a single sheaf; rather, it corresponds to a whole Gr-equivalence class sheaves. One may check, however, that any semi-stable sheaf Gr-equivalent to a rank 1, torsion-free sheaf is itself rank 1, torsion-free, so there is no ambiguity in the given definition.

We now turn our attention to showing that $\bar{J}_{L,d}(X/S)$ is closed-and-open in $M(\mathcal{O}_X, P_d)$. Consider the locus $Z \subseteq Q^\circ$ parametrizing quotients $q: \mathcal{O}_X(-b)^{\oplus r} \rightarrow F$ with the property that F is a rank 1, torsion-free sheaf on X . This is a SL_r -invariant subset that is closed and open in Q° ([Pan96, Lemma 8.1.1]), so its image must be closed-and-open in $M(\mathcal{O}_X, P_d)$ ([Sim94, Lemma 1.10] or [MFK94, pg. 8, Remark 6]). But, by definition, $\pi(Z)$ is just $\bar{J}_{L,d}(X/S)$.

We still need to show that $\bar{J}_{L,d}(X/S)$ universally co-represents $\bar{J}_{L,d}^\#(X/S)$. By universality, $\bar{J}_{L,d}(X/S)$ is the universal categorical quotient of Z by SL_r . An inspection of Simpson's proof shows that this scheme universally co-represents $\bar{J}_{L,d}^\#(X/S)$. This completes the proof. ■

2.7. The Oda-Seshadri Construction. Degree 0 compactified Jacobians associated to a nodal curve X were constructed by Oda and Seshadri in [OS79]. As in Simpson's work, their construction uses GIT, but the specifics are different. For example, they work with a slightly different open subscheme of a Quot scheme, they construct a different linearization (by embedding into a product of projective spaces), and they form the quotient in a different way (quotienting by GL_r rather than SL_r). Rather than delving into the details, we focus on reviewing the moduli-theoretic description of their compactified Jacobians.

The compactified Jacobians of Oda–Seshadri parameterize rank 1, torsion-free sheaves whose multi-degree satisfies a numerical condition, but in contrast to Simpson Jacobians, that condition is ϕ -semi-stability, not slope semi-stability. The parameter ϕ lies in a real vector space $\partial(C_1(\Gamma_X, \mathbb{R}))$ defined in terms of the combinatorics of the dual graph of X . Here we only recall the minimal amount about ϕ needed to define $\bar{J}_\phi(X)$. A complete description of ϕ would require a digression on the homology of graphs, and the reader is directed to [OS79] for more details. Here we define $\partial(C_1(\Gamma_X, \mathbb{R}))$ as follows. Write X_1, \dots, X_n for the components of X and v_1, \dots, v_n for the corresponding vertices of the dual graph Γ_X . By definition, $C_0(\Gamma_X, \mathbb{R})$ is the free \mathbb{R} -vector space with basis v_1, \dots, v_n , and $\partial(C_1(\Gamma_X, \mathbb{R}))$ is the subspace of $C_0(\Gamma_X, \mathbb{R})$ consisting of elements $\sum \phi_i v_i$ with $\sum \phi_i = 0$.

The ϕ -semi-stability condition on a rank 1, pure sheaf is defined in terms of a complex $\bar{K}(\Gamma_X)$ associated to the homology of Γ_X , but for line bundles, the condition admits a particularly simple description, and for the sake of exposition, we only review the definition of ϕ -semi-stability for line bundles. Later, we will relate ϕ -semi-stability to slope semi-stability, and the reader unfamiliar with ϕ -semi-stability for rank 1, pure sheaves can take that reformulation in terms of slopes as the definition (see Proposition 2.7).

Let $\phi \in \partial(C_1(\Gamma_X, \mathbb{R}))$ be given. If $Y \subset X$ is a subcurve, then let ϕ_Y denote the number

$$\phi_Y := \sum_{X_i \subset Y} \phi_i.$$

A degree 0 *line bundle* I is said to be **ϕ -semi-stable** if the inequality

$$(2.4) \quad \deg(I|_Y) \leq \phi_Y + \frac{\#(Y \cap Y^c)}{2}.$$

holds for all subcurves $Y \subset X$. Here Y^c is the closure of the complement of Y . If the inequality is always strict, then we say that the *line bundle* I is **ϕ -stable**. Inequality (2.4) is almost the same as [OS79, (**), pg. 8], but the inequalities differ due to the fact that the line bundle appearing in (**) has been tensored with a fixed ample line bundle. Let us indicate how (2.4) arises. The

definition of ϕ -stability is given on [OS79, pg. 51], and [OS79, Prop 11.1, pg. 51] states that I is ϕ -semi-stable if and only if the vector

$$\partial\bar{D}(I) := \sum \deg(I|_{X_i})v_i \in C_0(\Gamma_X, \mathbb{R})$$

lies in the polytope $\phi + \partial V_J(0)$. Corollary 6.3 of that paper [OS79, pg. 29] describes this polytope as an intersection of linear half-spaces, and Inequality (2.4) is just that description written out in different notation.

Later it will be convenient to rewrite Inequality (2.4) as a lower bound on $\deg(I|_Y)$, and this can be done easily. Because $\deg(I|_Y) = -\deg(I|_{Y^c})$ and $\phi_Y = -\phi_{Y^c}$, the inequality is equivalent to

$$(2.5) \quad \deg(I|_Y) \geq \phi_Y - \frac{\#(Y \cap Y^c)}{2}.$$

It will be easier to compare this second inequality to the inequalities arising from slope stability. In Proposition 2.7, we identify L -slope semi-stable sheaves with ϕ -semi-stable sheaves, and we define ϕ -poly-stability to be the condition that corresponds to slope poly-stability. This condition can be expressed in terms of ϕ , but we do not do so here.

We now define $\bar{J}_\phi^\sharp(X)$ and $\bar{J}_\phi(X)$. The **ϕ -compactified Jacobian functor**

$$\bar{J}_\phi^\sharp(X): k\text{-Sch} \rightarrow \text{Sets}$$

is the functor whose T -valued points are isomorphism classes of families of coherent sheaves that are fiber-wise rank 1, torsion-free, and ϕ -semi-stable. Oda and Seshadri have proven that $\bar{J}_\phi^\sharp(X)$ is co-represented by a projective scheme $\bar{J}_\phi(X)$ called the **ϕ -compactified Jacobian** ([OS79, Thm. 12.14, pg. 73]). The reader may have noticed that the functor studied in [OS79] is not $\bar{J}_\phi^\sharp(X)$, but rather the associated étale sheaf. However, the canonical natural transformation from $\bar{J}_\phi^\sharp(X)$ to the associated étale sheaf is a local isomorphism, so for questions of co-representability, the distinction is irrelevant.

2.8. The Caporaso–Pandharipande construction. The final compactified Jacobian we discuss is the universal compactified Jacobian. This scheme was first constructed by Caporaso, but again, we focus on Pandharipande’s later construction of the space as a family $\bar{J}_{d,g} \rightarrow \bar{M}_g$ over the coarse moduli space of stable curves that coarsely parameterizes slope semi-stable rank 1, torsion-free sheaves on stable curves.

To make this precise, suppose that we are given a k -scheme T and two pairs $(X_1/T, I_1)$, $(X_2/T, I_2)$ each consisting of a family X_j of genus g stable curves over T and a family I_j of slope semi-stable, rank 1, torsion-free sheaves. Here the slope semi-stability condition is taken with respect to the relative dualizing sheaf. We say that (X_1, I_1) is **equivalent** to (X_2, I_2) if there exists an isomorphism $\phi: X_1 \xrightarrow{\sim} X_2$ such that I_1 is isomorphic to $\phi^*(I_2)$. The **universal compactified Jacobian functor** of degree d is the functor

$$\bar{J}_{d,g}^\sharp: k\text{-Sch.} \rightarrow \text{Sets}$$

whose T -valued points are equivalence classes of pairs (X, I) , where I is required to have fiber-wise degree d . Pandharipande has proven that $\bar{J}_{d,g}^\sharp$ is co-representable by a projective scheme $\bar{J}_{d,g}$ called the **universal compactified Jacobian** ([Pan96, Thm. 8.3.2, 9.1.1]). Furthermore, he has constructed an isomorphism between $\bar{J}_{d,g}$ and the moduli space constructed by Caporaso ([Pan96, Thm. 10.3.1]).

The existence theorem we have stated is slightly different from the formulation in [Pan96], because $\bar{J}_{d,g}^\sharp$ is not the functor Pandharipande uses. He works with a functor parameterizing equivalence classes of slope semi-stable sheaves with respect to an equivalence relation coarser than the relation we have defined. He identifies (X_1, I_1) with (X_2, I_2) if there exists an isomorphism $\phi: X_1 \rightarrow X_2$

and a line bundle M on T such that I_1 is isomorphic to $\phi^*(I_2) \otimes p^*(M)$. Here $p: X \rightarrow T$ is the structure map. For questions of co-representability, the distinction is irrelevant because the tautological transformation from $\bar{J}_{d,g}$ to Pandharipande's functor is a local isomorphism.

Pandharipande constructs $\bar{J}_{d,g}$ as a GIT quotient of a relative Quot scheme. To begin, we may assume d is sufficiently large because tensoring with the dualizing sheaf defines a canonical isomorphism between $\bar{J}_{d,g}^\sharp$ and $\bar{J}_{d+2g-2,g}^\sharp$. Thus, let d be large and fixed. Set $N := 10(2g-2) - g$ and $e := 10(2g-2)$.

Inside of the Hilbert scheme of degree e curves in \mathbb{P}^N , we can consider the locally closed subscheme H_g parameterizing non-degenerate, 10-canonically embedded stable curves. The product $H_g \times \mathbb{P}^N$ contains the universal 10-canonically embedded curve X_g (denoted U_g in [Pan96]), and associated to this family is the relative Quot scheme

$$\begin{aligned} & \text{Quot}(\mathcal{O}_{X_g}^{\oplus r}, P_d(t)), \text{ where} \\ & P(t) = P_d(t) := e \cdot t + d + 1 - g, \\ & r := P(0). \end{aligned}$$

The product group $\text{SL}_r \times \text{SL}_{N+1}$ acts on this Quot scheme by making SL_r act on $\mathcal{O}_{X_g}^{\oplus r}$ by changing bases, SL_{N+1} act on \mathbb{P}^N by changing projective coordinates, and then making $\text{SL}_r \times \text{SL}_{N+1}$ act on the Quot scheme by the product action. As in Simpson's work, the action of $\text{SL}_r \times \text{SL}_{N+1}$ admits a natural linearization coming from the construction of the Quot scheme.

Inside of $\text{Quot}(\mathcal{O}_{X_g}^{\oplus r}, P_d)$ is the invariant closed-and-open subset parameterizing rank 1 quotients ([Pan96, Lemma 8.1.1]), and the universal Jacobian $\bar{J}_{d,g}$ is defined to be the associated GIT quotient. In a series of theorems [Pan96, Thm. 8.2.1, 9.1.1], Pandharipande proves that $\bar{J}_{d,g}$ co-represents $\bar{J}_{d,g}^\sharp$ and admits a suitable map $\bar{J}_{d,g} \rightarrow \bar{M}_g$.

As in Simpson's work, Pandharipande proves this by showing that GIT (semi-)stability of a quotient $q: \mathcal{O}_X^r \rightarrow I$ is equivalent to slope (semi-)stability of I with respect to $\omega^{\otimes 10}$ ([Pan96, Thm. 8.2.1]). He does so by showing that GIT (semi-)stability with respect to the action of $\text{SL}_r \times \text{SL}_{N+1}$ is equivalent to GIT (semi-)stability SL_r ([Pan96, Prop. 8.2.1]) and then proving that the second condition is equivalent to slope (semi-)stability ([Pan96, Thm. 2.1.1]). Pandharipande does not discuss poly-stability, but his argument also proves that GIT poly-stability coincides with slope poly-stability. Specifically, in the course of proving [Pan96, Prop. 8.2.1] he shows that the SL_r -orbit closures of two points meet if and only if the $\text{SL}_r \times \text{SL}_{N+1}$ -orbit closures meet. Thus, it is enough to prove that GIT poly-stability with respect to SL_r is equivalent to slope poly-stability, and this result is [Sim94, Thm. 1.21].

The proof Pandharipande gives actually establishes a slightly stronger result than is stated in the paper.

Lemma 2.4. *The scheme $\bar{J}_{d,g}$ uniformly co-represents $\bar{J}_{d,g}^\sharp$. If we further assume that $\text{char}(k) = 0$, then $\bar{J}_{d,g}$ universally co-represents $\bar{J}_{d,g}^\sharp$.*

Proof. This follows from the analogous facts about GIT quotients. First, we consider the uniformity statement. We may assume $d \gg 0$, and under this assumption, $\bar{J}_{d,g}$ is the quotient of the appropriate semi-stable locus Q^{ss} . Given a flat morphism $T \rightarrow \bar{J}_{d,g}$, let Q_T denote the pullback of Q^{ss} to T . Uniformity implies that T is the categorical quotient of Q_T by the natural action of $\text{SL}_r \times \text{SL}_{N+1}$, and one may check that the *proof* of [Pan96, Thm. 9.1.1] shows that Q_T co-represents $\bar{J}_{d,g}^\sharp \times_{\bar{J}_{d,g}} T$. If we additionally assume that $\text{char}(k) = 0$, then the categorical quotient of the Quot scheme is universal, so the argument just given remains valid when $T \rightarrow \bar{J}_{d,g}$ is arbitrary. ■

Consider now the fiber $\bar{J}_{d,g}|_x$ of $\bar{J}_{d,g} \rightarrow \bar{M}_g$ over a point $x \in \bar{M}_g$. This is a projective scheme that, by Lemma 2.4, has a natural moduli-theoretic interpretation (provided $\text{char}(k) = 0$): it co-represents $\bar{J}_{d,g}^\#|_x$. If x corresponds to a stable curve X , then $\bar{J}_{d,g}|_x$ can be considered as a compactified Jacobian associated to X . However, it does not enjoy all the properties one might expect of a compactified Jacobian. For example, there are examples where X is non-singular, but $\bar{J}_{d,g}|_x$ is not equal to the Jacobian of X . None the less, when $\text{char}(k) = 0$, we will call the fiber $\bar{J}_{d,g}$ the **Caporaso–Pandharipande compactified Jacobian** and denote it by $\bar{J}_d(X)$. The relationship between the Caporaso–Pandharipande compactified Jacobian and the other compactified Jacobians studied here is described by Fact 2.6 (3) below.

2.9. Relations among compactified Jacobians. There are several isomorphisms between the compactified Jacobians that we have just discussed, and we exhibit those isomorphisms here. Many of the isomorphism are constructed by manipulating co-representability properties. One exception is the identification of Simpson Jacobians with ϕ -compactified Jacobians, and for this we need to explicitly relate slope stability to ϕ -stability. This was done in [Ale04], but we recall the computation for the sake of completeness.

Suppose (X, L) is a pair consisting of a nodal curve X and an ample line bundle L . What does it mean for a rank 1, torsion-free sheaf I to be slope semi-stable with respect to L ? Recall from Fact 2.1 that I is slope semi-stable (with respect to L) if and only if $\mu_L(I) \leq \mu_L(i_*(I_Y))$ for all subcurves $i: Y \hookrightarrow X$. Both slopes can be computed explicitly. The polynomials $P_L(i_*(I_Y), t)$ and $P_{i^*(L)}(I_Y, t)$ are equal (by the projection formula), so the second computation reduces to the first. For a rank 1, torsion-free sheaf I on a curve, the Hilbert polynomial is

$$\begin{aligned} P(I, t) &= \deg(L) \cdot t + \chi(I) \\ &= \deg(L) \cdot t + \deg(I) + 1 - g. \end{aligned}$$

Thus, Inequality (2.2) can be rewritten as

$$\frac{\deg(I) - 1/2 \deg(\omega_X)}{\deg(L)} \leq \frac{\deg(I_Y) - 1/2 \deg(\omega_X|_Y) + 1/2 \#(Y \cap Y^c)}{\deg(L|_Y)},$$

or equivalently as

$$(2.6) \quad \deg(I_Y) \geq \frac{\deg(L|_Y)}{\deg(L)} \left(\deg(I) - \frac{\deg(\omega_X)}{2} \right) + \frac{\deg(\omega_X|_Y)}{2} - \frac{\#(Y \cap Y^c)}{2}.$$

Here Y^c again denotes the closure of the complement of Y . In deriving the above expression, we have used the relations

$$\begin{aligned} 1 - g &= \frac{-\deg(\omega_X)}{2}, \\ \omega_X|_Y &= \omega_Y(p_1 + \cdots + p_n), \text{ where } \{p_i\} = Y \cap Y^c. \end{aligned}$$

Fact 2.1 implies that I is slope semi-stable (with respect to L) if and only if (2.6) holds for all subcurves $Y \subset X$.

Specializing to the case $\deg(I) = 0$, the slope stability inequality (2.6) is a special case of (2.5). Indeed, suppose we label the irreducible components of X and the corresponding vertices of Γ_X as before. If we define

$$(2.7) \quad \phi_i := -\frac{\deg(\omega_X)}{2} \cdot \frac{\deg(L|_{X_i})}{\deg(L)} + \frac{\deg(\omega_X|_{X_i})}{2},$$

then these numbers satisfy $\sum \phi_i = 0$, and so $\phi := \sum \phi_i v_i \in \partial(\vec{C}_1(\Gamma_X, \mathbb{R}))$. When I is a line bundle, Inequality (2.6) is just the ϕ -semi-stability inequality (2.5). We have not explained ϕ -semi-stability

for sheaves that fail to be locally free, but a more careful study of [OS79] shows that this remains true when I fails to be locally free.

We may almost immediately conclude that every Simpson Jacobian of degree 0 is a ϕ -compactified Jacobian. Before stating this formally, we collect one more definition. The following group action arises when relating $\bar{J}_{L,d}(X)$ to $\bar{J}_{d,g}$.

Definition 2.5. If X is a nodal curve, then the **natural action** of $\text{Aut}(X)$ on $\bar{J}_{L,d}^\#(X)$ is the action defined by making a T -valued point g of $\text{Aut}(X)$ act on the set of T -valued points of $\bar{J}_{L,d}^\#(X)$ by sending the isomorphism class of a sheaf I on X_T to the isomorphism class of $g_*(I)$. The **natural action** of $\text{Aut}(X)$ on $\bar{J}_{L,d}(X)$ is the unique action making $\bar{J}_{L,d}^\#(X) \rightarrow \bar{J}_{L,d}(X)$ equivariant (which exists by the universal property of co-representability).

Assume X is stable, so that $\text{Aut}(X)$ is a finite group. In this case the categorical quotient $\bar{J}_{L,d}(X)/\text{Aut}(X)$ exists and co-represents the quotient functor $\bar{J}_{L,d}^\#(X)/\text{Aut}(X)$. This scheme appears as a fiber of $\bar{J}_{d,g} \rightarrow \bar{M}_g$ (in characteristic 0).

Fact 2.6 (Alexeev [Ale04]). *Assume $k = \mathbb{C}$. Then we have the following identifications:*

- (1) *Let (X, L) be a polarized nodal curve. If ϕ is defined by Eqn. (2.7), then $\bar{J}_{L,0}^\#(X)$ equals $\bar{J}_\phi^\#(X)$. In particular, there is a canonical isomorphism*

$$\bar{J}_{L,0}(X) \xrightarrow{\sim} \bar{J}_\phi(X).$$

- (2) *The functor $\bar{J}_{\omega,d}^\#(\mathcal{C}_g/\bar{M}_g^\circ)$ equals $\bar{J}_{d,g}^\#|_{(\bar{M}_g^\circ)^\#}$, where $\mathcal{C}_g \rightarrow \bar{M}_g^\circ$ is the universal family over \bar{M}_g° and ω is the relative dualizing sheaf. In particular, there is a canonical isomorphism*

$$\bar{J}_{\omega,d}(\mathcal{C}_g/\bar{M}_g^\circ) \xrightarrow{\sim} \bar{J}_{d,g}|_{\bar{M}_g^\circ}.$$

- (3) *Let $x \in \bar{M}_g$ correspond to a stable curve X , and set ω equal to the dualizing sheaf. Then there is a tautological transformation*

$$\bar{J}_{\omega,d}^\#(X)/\text{Aut}(X) \rightarrow \bar{J}_{d,g}^\#|_x$$

that is a local isomorphism. In particular, there is a canonical isomorphism

$$\bar{J}_{\omega,d}(X)/\text{Aut}(X) \xrightarrow{\sim} \bar{J}_{d,g}|_x.$$

Proof. With the work we have done, this is an exercise in co-representability. For Item (1), we have just explained why slope (semi-)stability with respect to L is equivalent to ϕ -semi-stability, and the equality $\bar{J}_{L,0}^\#(X) = \bar{J}_\phi^\#(X)$ is just another way of stating that. The existence of a canonical isomorphism between the coarse spaces is then a consequence of Yoneda's lemma.

Item (2) is even easier to prove. An inspection of the relevant definitions shows $\bar{J}_{\omega,d}^\#(\mathcal{C}_g/\bar{M}_g^\circ) = \bar{J}_{d,g}^\#|_{(\bar{M}_g^\circ)^\#}$. By Lemma 2.4, the coarse space $\bar{J}_{d,g}|_{\bar{M}_g^\circ}$ co-represents $\bar{J}_{d,g}^\#|_{(\bar{M}_g^\circ)^\#}$, so the existence of a canonical isomorphism between coarse spaces follows immediately.

Finally, we need to prove Item (3). Unwinding definitions, we see that the T -valued points of $\bar{J}_{\omega,d}^\#(X)$ correspond to equivalence classes of families of rank 1, torsion-free sheaves on the constant family $X \times T$, while the T -valued points of $\bar{J}_{d,g}^\#|_x$ correspond to equivalence classes of families of rank 1, torsion-free sheaves on an iso-trivial family of curves with fiber X . Thus, the functors $\bar{J}_{\omega,d}^\#(X)$ and $\bar{J}_{d,g}^\#|_x$ are not equal, but there is a tautological transformation $\bar{J}_{\omega,d}^\#(X)/\text{Aut}(X) \rightarrow \bar{J}_{d,g}^\#|_x$, which is easily seen to be a local isomorphism. As before, the existence of an isomorphism between coarse moduli schemes now follows formally. \blacksquare

In stating Fact 2.6, we have been somewhat conservative in assuming that $k = \mathbb{C}$. The construction of the isomorphisms in the first two items does not make use of any assumptions on k , and we assume $k = \mathbb{C}$ only because Simpson requires this condition in his paper. We do, however, use that $k = \mathbb{C}$ in our proof of Item 3. Specifically, the proof uses the fact that $\bar{J}_{d,g}$ universally corepresents $\bar{J}_{d,g}^\#$ (Lemma 2.4), which is an easy consequence of the fact that $\bar{J}_{d,g}$ is *universal* categorical quotient of the Quot scheme. Universality is automatic in characteristic zero, but in general can fail to hold in positive characteristic. It would be interesting to know if this failure occurs in the construction of $\bar{J}_{d,g}$.

We should also indicate what results do not follow from Fact 2.6 and are in fact false. The fact states that every degree 0 Simpson Jacobian is a ϕ -compactified Jacobian, but the converse does not hold. **There are ϕ -compactified Jacobians that do not arise as degree 0 Simpson Jacobians.** An explicit example is provided by the genus 2 curve X that consists of two rational components meeting in three nodes. If $\phi = \phi_1 \cdot v_1 + \phi_2 \cdot v_2$, then one can compute that $\bar{J}_\phi(X)$ has two irreducible components if $\phi_1, \phi_2 \in 1/2 + \mathbb{Z}$ and three irreducible components otherwise. Given an ample line bundle L with bidegree (a, b) , the associated ϕ -parameter is

$$\phi = (-1/2 + b/(a+b)) \cdot v_1 + (-1/2 + a/(a+b)) \cdot v_2,$$

and the coefficients never differ from $1/2$ by integers (because $a, b > 0$).

One can, however, obtain additional ϕ -compactified Jacobians from Simpson Jacobians as follows. Given an auxiliary line bundle M of degree d , one can identify $\bar{J}_{L,d}(X)$ with a moduli space $\bar{J}_{L,d}^M(X)$ of degree 0 line bundles by translating via M^\vee . The space $\bar{J}_{L,d}^M(X)$ is canonically isomorphic to $\bar{J}_\phi(X)$, where ϕ is defined by labeling the irreducible components of X and setting

$$(2.8) \quad \phi_i := -\deg(M|_{X_i}) + \frac{1}{2} \deg(\omega_X|_{X_i}) + (d - \frac{1}{2} \deg(\omega_X)) \frac{\deg(L|_{X_i})}{\deg(L)}.$$

Varying over all ample line bundles L , all degrees $d \in \mathbb{Z}$ and all line bundles M of degree d , we obtain every element of $\partial(\vec{C}_1(\Gamma_X, \mathbb{Q}))$. Indeed, given $\phi \in \partial(\vec{C}_1(\Gamma_X, \mathbb{Q}))$ we can certainly find a line bundle M of sufficiently large total degree $d > g - 1$ such that for every i

$$a_i := \phi_i + \deg(M|_{X_i}) - \frac{1}{2} \deg(\omega_X|_{X_i}) \in \mathbb{Q}_{>0}.$$

Similarly, we can find a sufficiently divisible natural number $e \in \mathbb{N}$ such that for every i

$$b_i := e \frac{a_i}{d + g - 1} \in \mathbb{Z}_{>0}.$$

Finally, by taking an ample line bundle L of total degree e such that $\deg_{X_i} L = b_i$, we have found L and M as before for which Eqn. (2.8) is satisfied. Furthermore, if $\phi \in \partial(\vec{C}_1(\Gamma_X, \mathbb{R}))$, then the ϕ -semi-stability condition remains unchanged if we replace ϕ with a sufficiently close rational approximation, and so **for every ϕ -compactified Jacobian $\bar{J}_\phi(X)$ there is an integer d and line bundles $L, M \in \text{Pic}(X)$ with L ample such that $\bar{J}_\phi(X) \cong \bar{J}_{L,d}^M(X) \cong \bar{J}_{L,d}(X)$.**

Similarly, we can extend the definition of ϕ -compactified Jacobians by simply allowing ϕ to be an element of $C_0(\Gamma_X, \mathbb{R})$ satisfying $\sum \phi_i = d \in \mathbb{Z}$. Given such a ϕ , $\bar{J}_\phi(X)$ is then a coarse moduli space of degree d sheaves, and with this extended definition, **every Simpson Jacobian of degree d is a ϕ -compactified Jacobian of degree d** (use Eqn. (2.8) with M equal to the trivial line bundle). As was already observed for $d = 0$, **there are ϕ -compactified Jacobians of degree d that are not Simpson Jacobians of degree d** . The most extreme case is $d = g - 1$. As was observed in [Ale04], there is a unique Simpson Jacobian of this degree, but there are typically many ϕ -compactified Jacobians. Finally we note that an argument similar to the one just given

shows that **given a ϕ -compactified Jacobian $\bar{J}_\phi(X)$ of degree d , there is an integer e and an ample line bundle $L \in \text{Pic}(X)$ such that $\bar{J}_\phi(X) \cong \bar{J}_{L,e}(X)$.**

To summarize:

Proposition 2.7. *Let X be a nodal curve and let $\phi \in \partial(C_1(\Gamma_X, \mathbb{R}))$. There exists an integer d and line bundles L, M on X with L ample such that the rule $I \mapsto I \otimes M$ defines an isomorphism*

$$\bar{J}_\phi^\sharp(X) \xrightarrow{\sim} \bar{J}_{L,d}^\sharp(X)$$

that induces an isomorphism on compactified Jacobians. Moreover, a rank 1, torsion-free sheaf I on X is ϕ -stable (resp. poly-stable, semi-stable) if and only if $I \otimes M$ is slope stable (resp. poly-stable, semi-stable) with respect to L .

Proof. This follows from what is above. ■

2.10. Comparison of constructions. We conclude with an explanation of how the results from this section are used later in this paper. Our main results describe the local structure of the Oda-Seshadri, Simpson, and universal compactified Jacobians. Proposition 2.7 allows us to restrict attention to the latter two spaces. The central step in the proof of Theorem A is Theorem 6.1. To establish this theorem, we make use of the GIT constructions of the spaces. Specifically, we will need to assert that the spaces can be constructed as GIT quotients of an open subscheme of a (possibly relative) Quot scheme parameterizing quotients $q: \mathcal{O}_X^{\oplus r} \rightarrow I$ with the property that

- (1) $H^1(X, I) = 0$;
- (2) $q: H^0(X, \mathcal{O}_X^{\oplus r}) \rightarrow H^0(X, I)$ is an isomorphism;
- (3) I is a rank 1, torsion-free sheaf.

Due to the existence of special linear series, conditions (1) and (2) obviously can not be expected to hold for every slope semi-stable sheaf satisfying (3). However, in each construction, one can avoid this problem by tensoring by a sufficiently high power of the relatively ample line bundle. The new moduli space obtained in this way will be canonically isomorphic to the original moduli space, but will parameterize sheaves satisfying (1)-(3). This was already discussed in the section on the Pandharipande construction, and is essentially used in the Simpson construction as well, where conditions (1) and (2) are replaced by similar conditions on a twist of I by $\mathcal{O}_X(1)$.

3. DEFORMATION THEORY

In the previous section, we studied the representability properties of global moduli functors parameterizing rank 1, torsion-free sheaves on a nodal curve. This section focuses on the analogous local topic: the pro-representability properties of deformation functors parameterizing infinitesimal deformations of a fixed rank 1, torsion-free sheaf. The main result is Corollary 3.17, which explicitly describes miniversal deformations rings parameterizing such deformations. The corollary is used in §6 to prove Theorem A by relating a deformation ring to the completed local ring of a compactified Jacobian (Thm. 6.1).

3.1. The deformation functors. We begin by reviewing the deformation functors of interest. The reader who has read §2 should note that, while stability conditions played a key role in the definition of the global moduli functors, they play no role in the definition of the local deformation functors. Indeed, (semi-)stability is a fibral condition, so a family of rank 1, torsion-free sheaves over the spectrum $\text{Spec}(A)$ of an artin local k -algebra is a family of (semi-)stable sheaves if and only if the fiber over the unique closed point is (semi-)stable.

Definition 3.1. Suppose we are given a k -scheme S , a finitely presented \mathcal{O}_S -module F , and a local k -algebra A with residue field k . A **deformation of the pair (S, F)** over A is a quadruple (S_A, F_A, i, j) that consists of

- (1) a flat A -scheme S_A ;
- (2) a A -flat, finitely presented \mathcal{O}_{S_A} -module F_A ;
- (3) an isomorphism $i: S_A \otimes_A k \xrightarrow{\sim} S$;
- (4) an isomorphism $j: i_*(F_A \otimes_A k) \xrightarrow{\sim} F$ of \mathcal{O}_S -modules.

The **trivial deformation** of a pair (S, F) over A is defined to be the quadruple $(S \otimes_k A, F \otimes_k A, A, i_{\text{can}}, j_{\text{can}})$. Here i_{can} and j_{can} are defined to be the canonical maps. If (S'_A, F'_A, i', j') is a second deformation of the pair (S, F) , then an **isomorphism** from (S_A, F_A, i, k) to (S'_A, F'_A, i', j') is defined to be a pair (ϕ, ψ) that consists of

- (1) an isomorphism $\phi: S_A \xrightarrow{\sim} S'_A$ over A ;
- (2) an isomorphism $\psi: \phi_*(F_A) \xrightarrow{\sim} F'_A$ of $\mathcal{O}_{S_{A'}}$ -modules.

We require that the isomorphisms ϕ and ψ are compatible in the sense that the following two diagrams commute:

$$(3.1) \quad \begin{array}{ccc} S_A \otimes_A k & \xrightarrow{\phi \otimes 1} & S'_A \otimes_A k \\ \downarrow i & & \downarrow i' \\ S & \xrightarrow{\text{Id}} & S; \end{array}$$

$$(3.2) \quad \begin{array}{ccc} i_*(F_A \otimes_A k) & \xrightarrow{i'_*(\psi \otimes 1)} & i'_*(F'_A \otimes_A k) \\ \downarrow j & & \downarrow j' \\ F & \xrightarrow{\text{Id}} & F, \end{array}$$

where the top horizontal map of (3.2) is well defined since $i_*(F_A \otimes_A k) = i'_*(\phi_* F_A \otimes_A k)$ by virtue of diagram (3.1).

A **deformation of the scheme S** over A is defined by omitting the data of F_A and j from the definition of a deformation of a pair. Similarly, an **isomorphism** from one deformation (S_A, i) of S to another (S'_A, i') is defined by omitting ψ from Definition 3.1. The scheme S always admits the **trivial deformation** over A given by the pair $(S \otimes_k A, i_{\text{can}})$.

A **deformation of a sheaf F** over A is defined to be a pair (F_A, j) such that the quadruple $(S \otimes_k A, F_A, i_{\text{can}}, j)$ is a deformation of the pair (S, F) . An **isomorphism** from one deformation of F to another is defined to be a deformation of the associated deformations of the pair (S, F) . The trivial deformation of the pair (S, F) may be considered as a **trivial deformation** of F .

Let Art_k be the category of artin local k -algebras with residue field k . Recall that a **deformation functor** is a functor $F: \text{Art}_k \rightarrow \text{Sets}$ of artin rings with the property that $F(k)$ is a singleton set. We study the following deformation functors.

Definition 3.2. Define functors $\text{Def}_S, \text{Def}_F, \text{Def}_{(S,F)}: \text{Art}_k \rightarrow \text{Sets}$ by

$$(3.3) \quad \begin{aligned} \text{Def}_{(S,F)}(A) &:= \{\text{iso. classes of deformations of } (S, F) \text{ over } A\}, \\ \text{Def}_S(A) &:= \{\text{iso. classes of deformations of } S \text{ over } A\}, \\ \text{Def}_F(A) &:= \{\text{iso. classes of deformations of } F \text{ over } A\}. \end{aligned}$$

The automorphism groups $\text{Aut}(S, F)$, $\text{Aut}(S)$, and $\text{Aut}(F)$ act on appropriate deformations functors, and this action will be studied in §4. The reader should be familiar with the definitions of $\text{Aut}(S)$ and $\text{Aut}(F)$, but perhaps not of $\text{Aut}(S, F)$.

Definition 3.3. An **automorphism** of (S, F) is a pair (σ, τ) that consists of:

- (1) an automorphism $\sigma: S \xrightarrow{\sim} S$;

(2) an isomorphism of sheaves $\tau : \sigma_* F \xrightarrow{\sim} F$.

The group of automorphisms of (S, F) , denoted by $\text{Aut}(S, F)$, fits into the exact sequence

$$(3.4) \quad 0 \rightarrow \text{Aut}(F) \rightarrow \text{Aut}(S, F) \rightarrow \text{Aut}(S) \\ (\sigma, \tau) \mapsto \sigma.$$

These automorphism groups act naturally on their respective functors.

Definition 3.4. Let (S, F) be a given pair. Then we define the **natural action** of

- $\text{Aut}(S, F)$ on $\text{Def}_{(S, F)}$ by making an element $(\sigma, \tau) \in \text{Aut}(S, F)$ acts as

$$(S_A, F_A, i, j) \mapsto (S_A, F_A, \sigma \circ i, \tau \circ \sigma_*(j)).$$

Here $\tau \circ \sigma_*(j)$ is the composition $\sigma_* i_*(F_A \otimes_A k) \xrightarrow{\sigma_*(j)} \sigma_*(F) \xrightarrow{\tau} F$;

- $\text{Aut}(S)$ on Def_S by making an element $\sigma \in \text{Aut}(S)$ acts as

$$(S_A, i) \mapsto (S_A, \sigma \circ i);$$

- $\text{Aut}(F)$ on Def_F by making an element $\tau \in \text{Aut}(F)$ acts as

$$(F_A, j) \mapsto (F_A, \tau \circ j).$$

Later we will relate the above deformation functors to the Quot scheme, so it is convenient to introduce the deformation functors arising from the Quot scheme. To avoid irrelevant foundational issues, we only define the deformation functors associated to nodal curves.

Definition 3.5. Let X be a nodal curve; F a coherent sheaf on X ; and $q : \mathcal{O}_X^r \twoheadrightarrow F$ a surjection. A **deformation of the pair** (X, q) over $A \in \text{Art}_k$ is a quadruple (X_A, i, q_A, j) where $q_A : \mathcal{O}_{X_A}^r \twoheadrightarrow F_A$ is a surjection such that (X_A, F_A, i, j) is a deformation of (X, F) in the sense of Definition 3.1. Furthermore, we require that the isomorphism $j : i_*(F_A \otimes_A k) \xrightarrow{\sim} F$ respects quotient maps, in the sense that $q = j \circ i_*(q_A \otimes 1)$.

Given a second deformation (X'_A, i', q'_A, j') of (X, q) over A , an **isomorphism** from (X_A, i, q_A, j) to (X'_A, i', q'_A, j') is defined to be a pair (ϕ, ψ) consisting of

- (1) an isomorphism $\phi : X_A \xrightarrow{\sim} X'_A$ over A ;
- (2) an isomorphism $\psi : \phi_*(F_A) \xrightarrow{\sim} F'_A$ of $\mathcal{O}_{A'}$ -modules.

The isomorphisms ϕ and ψ are required to fit into the following commutative diagram:

$$(3.5) \quad \begin{array}{ccc} \phi_*(\mathcal{O}_{X_A}^r) & \xrightarrow{\phi_*(q_A)} & \phi_*(F_A) \\ \downarrow \text{Id} & & \downarrow \psi \\ \mathcal{O}_{X'_A}^r & \xrightarrow{q'_A} & F'_A. \end{array}$$

A **deformation of q** over $A \in \text{Art}_k$ is defined to be a deformation of (X, q) of the form $(X \otimes_k A, i_{\text{can}}, q_A, j)$, where $(X \otimes_k A, i_{\text{can}})$ is the trivial deformation. An **isomorphism** from one deformation of q to another is defined to be an isomorphism of the associated deformations of (X, q) .

The deformation functors Def_q and $\text{Def}_{(X, q)}$ are defined in the expected manner.

Definition 3.6. We define functors $\text{Def}_q, \text{Def}_{(X, q)} : \text{Art}_k \rightarrow \text{Sets}$ by

$$(3.6) \quad \begin{aligned} \text{Def}_{(X, q)}(A) &:= \{\text{iso. classes of deformations of } (X, q) \text{ over } A\}, \\ \text{Def}_q(A) &:= \{\text{iso. classes of deformations of } q \text{ over } A\}. \end{aligned}$$

To study $\bar{J}_{d, g}$, we also need a slight generalization of $\text{Def}_{(X, q)}$.

Definition 3.7. Suppose that X is a stable curve; F a coherent sheaf; $q: \mathcal{O}_X^r \twoheadrightarrow F$ a quotient map; and $p: X \hookrightarrow \mathbb{P}^N$ is a 10-canonical embedding. A **deformation of the pair** (p, q) over $A \in \text{Art}_k$ is a quadruple (p_A, i, q_A, j) , where $p_A: X_A \hookrightarrow \mathbb{P}_A^N$ is closed embedding and (X_A, i, q_A, j) is a deformation of the pair (X, q) . We further require

- the line bundles $\mathcal{O}_{X_A}(1)$ and $\omega_{X_A/A}^{\otimes 10}$ are isomorphic;
- $p_A \otimes 1 = p \circ i$.

Given a second deformation (p'_A, i', q'_A, j') of (p, q) , we define an **isomorphism** from the first deformation to the second to be an isomorphism (ϕ, ψ) of the associated deformations of (X, q) with the property that

$$p_A = p'_A \circ \phi.$$

Definition 3.8. Define the functor $\text{Def}_{(p,q)}: \text{Art}_k \rightarrow \text{Sets}$ by

$$\text{Def}_{(p,q)}(A) := \{\text{iso. classes of deformations of } (p, q) \text{ over } A\}.$$

Note that there are forgetful transformations $\text{Def}_q \rightarrow \text{Def}_F$ and $\text{Def}_{(p,q)} \rightarrow \text{Def}_{(X,q)} \rightarrow \text{Def}_{(X,F)}$ that are formally smooth once F is sufficiently positive (see Lemma 6.3).

The deformation functors we study are parameterized by complete local k -algebras. There are several different ways in which a complete local k -algebra can parameterize a deformation functor. As in §2, if $A \in \text{Art}_k$ is an artin local k -algebra, then Eqn. (2.1) defines a functor $\text{Art}_k \rightarrow \text{Sets}$, and we say that a deformation functor Def is **representable** if it is isomorphic to the functor associated to such an A . However, with this definition, almost all deformation functors of interest fail to be representable. A weaker condition to impose is pro-representability. To a complete local k -algebra R with residue field k , we can associate the functor $\text{Art}_k \rightarrow \text{Sets}$ defined by

$$(3.7) \quad A \mapsto \text{Hom}_{\text{loc}}(R, A),$$

and a modification of the proof of Yoneda's Lemma shows that R can be recovered from this associated functor. We will write $\text{Spf}(R)$ for the functor defined by Eqn. (3.7). The notation is intended to suggest that $\text{Spf}(R)$ is the formal spectrum of R .

We say that a functor $\text{Def}: \text{Art}_k \rightarrow \text{Sets}$ is **pro-representable** if it is isomorphic to $\text{Spf}(R)$ for some complete local k -algebra with residue field k . A pair (R, π) consisting of such an algebra R and an isomorphism $\pi: \text{Spf}(R) \xrightarrow{\sim} \text{Def}$ is said to be a **deformation ring** for Def . An exercise in unraveling definitions shows that the completed local ring of an appropriate Quot scheme is a deformation ring for Def_q , and similarly for $\text{Def}_{(X,q)}$.

The functors Def_F and $\text{Def}_{(X,F)}$ are not always pro-representable, but do satisfy the weaker condition of admitting a miniversal deformation ring. Suppose that we are given a pair (R, π) consisting of a complete local k -algebra R and a natural transformation $\pi: \text{Spf}(R) \rightarrow \text{Def}$. We say that (R, π) is a **versal deformation ring** for Def if π is formally smooth. If π has the additional property that it induces an isomorphism on tangent spaces, then we say that (R, π) is a **miniversal deformation ring**. One can show that if (R, π) and (R', π') are both miniversal deformation rings for Def , then R is isomorphic to R' , but in contrast to the situation for deformation rings, there is no distinguished isomorphism $R \cong R'$. We now proceed to construct miniversal deformation rings for Def_I and $\text{Def}_{(X,I)}$.

3.2. The miniversal deformation rings. The existence of miniversal deformation rings for Def_I and $\text{Def}_{(X,I)}$ can be deduced from theorems of Schlessinger, but for later computations, we will want an explicit description of these rings. We derive such a description by relating Def_I and $\text{Def}_{(X,I)}$ to the analogous deformation functors associated to the node \mathcal{O}_0 . We begin by fixing some notation for the node.

Definition 3.9. The **standard node** \mathcal{O}_0 is the complete local k -algebra $k[[x, y]]/(xy)$. The **normalization** of the standard node is denoted $\tilde{\mathcal{O}}_0$.

As a subring of the total ring of fractions $\text{Frac}(\mathcal{O}_0)$, the normalization of \mathcal{O}_0 is equal to $\tilde{\mathcal{O}}_0 = \mathcal{O}_0[x/(x+y)]$. It follows that the quotient $\tilde{\mathcal{O}}_0/\mathcal{O}_0$ is a 1-dimensional k -vector space spanned by the image of $x/(x+y)$. Recall that $\tilde{\mathcal{O}}_0$ is also isomorphic to the ring $k[[x]] \oplus k[[y]]$, and the inclusion $\mathcal{O}_0 \rightarrow \tilde{\mathcal{O}}_0$ factors as

$$\frac{k[[x, y]]}{(xy)} \rightarrow k[[x]] \oplus k[[y]] \xrightarrow{\sim} \frac{k[[x, y]]}{(xy)} \left[\frac{x}{x+y} \right]$$

where the first map is given by $h(x, y) \mapsto (h(x, 0), h(0, y))$ and the second map is given by $(f, g) \mapsto (fx + gy)/(x + y)$.

Over \mathcal{O}_0 , there are exactly two rank 1, torsion-free modules up to isomorphism: the free module and a unique module that fails to be locally free. A proof of this statement can be found in [D'S79], where it is deduced from [Vas68, Thm. 3.1]. There are several ways to describe the module that fails to be locally free.

Definition 3.10. The unique rank 1, torsion-free module I_0 over \mathcal{O}_0 that fails to be locally free can be described as any one of the following modules:

- (1) the ideal $(x, y) \subset \mathcal{O}_0$, considered as an \mathcal{O}_0 -module,
- (2) the extension $\tilde{\mathcal{O}}_0 \supset \mathcal{O}_0$, considered as an \mathcal{O}_0 -module,
- (3) the \mathcal{O}_0 -module with presentation $\langle e, f : y \cdot e = x \cdot f = 0 \rangle$.

An isomorphism from the 3rd module to the 1st module is given by $e \mapsto x, f \mapsto y$, while an isomorphism from the 3rd to the 2nd is given by $e \mapsto x/(x+y), f \mapsto y/(x+y)$. In passing from one model of I_0 to another, we will always implicitly identify the modules via these specific isomorphisms.

3.2.1. Formal smoothness and reduction to the case of nodes. If I is a rank 1, torsion-free sheaf on a nodal curve X , then the study of Def_I and $\text{Def}_{(X, I)}$ reduces to the study of Def_{I_0} and $\text{Def}_{(\mathcal{O}_0, I_0)}$. Indeed, say that Σ is the set of nodes where I fails to be locally free. For a given $e \in \Sigma$, let X_e denote the spectrum of the completed local ring $\hat{\mathcal{O}}_{X, e}$ and I_e the pullback of I to X_e . There are forgetful transformations relating global deformations to local deformations:

$$(3.8) \quad \begin{aligned} \text{Def}_{(X, I)} &\rightarrow \prod_{e \in \Sigma} \text{Def}_{(X_e, I_e)}, \\ \text{Def}_I &\rightarrow \prod_{e \in \Sigma} \text{Def}_{I_e}, \\ \text{Def}_X &\rightarrow \prod_{e \in \Sigma} \text{Def}_{X_e}. \end{aligned}$$

All of these transformations are formally smooth. Indeed, for the last transformation, this is [DM69, Prop. 1.5]. That result together with [FGvS99, A.1-4] shows that the first transformation is formally smooth. Essentially the same argument also shows that the middle transformation is formally smooth, and this is a special case of [FGvS99, B.1].

We now construct deformation rings for Def_{I_0} and $\text{Def}_{(\mathcal{O}_0, I_0)}$. We begin by parameterizing deformations of (\mathcal{O}_0, I_0) .

Definition 3.11. Define a complete local k -algebra $S_2 = S_2(\mathcal{O}_0, I_0)$ by

$$S_2 := k[[t, u, v]]/(uv - t).$$

The algebraic deformation $(\mathcal{O}_{S_2}, i, I_{S_2}, j)$ of (\mathcal{O}_0, I_0) over S_2 is defined by setting

- $\mathcal{O}_{S_2} := S_2[[x, y]]/(xy - t)$;

- $i: \mathcal{O}_{S_2} \otimes_{S_2} k \xrightarrow{\sim} \mathcal{O}_0$ equal to the isomorphism that is the identity on the variables x and y ;
 - I_{S_2} equal to the \mathcal{O}_{S_2} -module with presentation
- $$(3.9) \quad I_{S_2} := \langle \tilde{e}, \tilde{f} : y \cdot \tilde{e} = -u \cdot \tilde{f}, x \cdot \tilde{f} = -v \cdot \tilde{e} \rangle;$$
- $j: i_*(I_{S_2} \otimes_{S_2} k) \xrightarrow{\sim} I_0$ equal to the isomorphism given by rules $\tilde{e} \otimes 1 \mapsto e$ and $\tilde{f} \otimes 1 \mapsto f$.

Deformations of I_0 alone are parameterized similarly.

Definition 3.12. Define $S_1 = S_1(I_0) := k[[u, v]]/(uv)$. The algebraic deformation (I_{S_1}, j) of I_0 over S_1 is defined by setting

- $\mathcal{O}_{S_1} = S_1[[x, y]]/(xy)$;
 - I_{S_1} equal to the \mathcal{O}_{S_1} -module with presentation
- $$(3.10) \quad \mathcal{I} := \langle \tilde{e}, \tilde{f} : y \cdot \tilde{e} = -u \cdot \tilde{f}, x \cdot \tilde{f} = -v \cdot \tilde{e} \rangle;$$
- $j: i_*(I_{S_1} \otimes_{S_1} k) \xrightarrow{\sim} I_0$ equal to isomorphism given by rules $\tilde{e} \otimes 1 \mapsto e$ and $\tilde{f} \otimes 1 \mapsto f$.

Remark 3.13. It may be more intuitive to describe the deformations in geometric terms. There is a versal deformation (resp. trivial deformation) $\mathcal{X} \rightarrow B$ of the node, with base

$$B = \operatorname{Spec} k[u, v, t]/(uv - t) \quad (\text{resp.} \quad B = \operatorname{Spec} k[u, v]/(uv))$$

and total space

$$\mathcal{X} = B \times \operatorname{Spec} k[x, y]/(xy - t) \quad (\text{resp.} \quad \mathcal{X} = B \times \operatorname{Spec} k[x, y]/(xy)).$$

The module I_{S_2} (resp. I_{S_1}) is essentially the “universal” ideal $I = (x - u, y - v) \subseteq \Gamma(\mathcal{X}, \mathcal{O}_X)$ considered as a module as in Definition 3.10 (3).

Lemma 3.14. S_2 is a miniversal deformation ring for $\operatorname{Def}_{(\mathcal{O}_0, I_0)}$. More precisely, the algebraic deformation $(\mathcal{O}_{S_2}, i, I_{S_2}, j)$ defines a transformation $\operatorname{Spf}(S_2) \rightarrow \operatorname{Def}_{(\mathcal{O}_0, I_0)}$ that realizes S_2 as the miniversal deformation ring for Def_I . Similarly, S_1 is a miniversal deformation ring for Def_{I_0} .

Proof. The claim concerning the ring S_1 was established in the course of proving Proposition 2.6 of [CMK]. The same argument holds for S_2 provided that one replaces the standard irreducible, nodal plane cubic used in that proof with a general pencil containing such a curve. ■

Given a rank 1, torsion-free sheaf I that fails to be locally free at a set of nodes Σ , there is a simple relation between Def_I and $\prod_{e \in \Sigma} \operatorname{Def}_{I_e}$.

Definition 3.15. Let $\operatorname{Def}_I^{\text{lt}} \subset \operatorname{Def}_I$ be the subfunctor parameterizing deformations that map to the trivial deformation under $\operatorname{Def}_I \rightarrow \prod_{e \in \Sigma} \operatorname{Def}_{I_e}$. Define $\operatorname{Def}_{(X, I)}^{\text{lt}}$ similarly. Elements of these deformation functors (valued in a given ring) are called **locally trivial** deformations (over that ring).

Lemma 3.16. Let X be a nodal curve; Σ a set of nodes; $g: X_\Sigma \rightarrow X$ the map that normalizes the nodes Σ ; and $I := g_*(L)$ the direct image of a line bundle L on X_Σ . Then the rule

$$(3.11) \quad (L, i) \mapsto (g_*(L), g_*(i))$$

defines an isomorphism $\operatorname{Def}_L \cong \operatorname{Def}_I^{\text{lt}}$.

Proof. The map $\operatorname{Def}_L \rightarrow \operatorname{Def}_I$ defined by Eqn. (3.11) has the property that the composition $\operatorname{Def}_L \rightarrow \operatorname{Def}_I \rightarrow \prod_{e \in \Sigma} \operatorname{Def}_{I_e}$ is the trivial map, so there is an induced map $\operatorname{Def}_L \rightarrow \operatorname{Def}_I^{\text{lt}}$. Studying the map $\operatorname{Def}_I \rightarrow \prod \operatorname{Def}_{I_e}$ and the associated map on tangent-obstruction theories, one can show that $\operatorname{Def}_I^{\text{lt}}$ is formally smooth with tangent space $T_1 = H^1(\underline{\operatorname{End}}(I))$. This vector space is just $H^1(X_\Sigma, \mathcal{O}_{X_\Sigma})$, which can be identified with the tangent space to Def_L in such a way that $T_1(\operatorname{Def}_L) \rightarrow T_1(\operatorname{Def}_I^{\text{lt}})$ is the identity. By formal smoothness, it follows that $\operatorname{Def}_L \rightarrow \operatorname{Def}_I^{\text{lt}}$ is an isomorphism. ■

Let us denote by R_1 the miniversal deformation ring of Def_I and by R_2 the miniversal deformation ring of $\text{Def}_{(X,I)}$ (which exists by, say, [FGvS99, § A]). Lemma 3.14 together with the discussion following Eqn. (3.8) allows us to describe the miniversal deformation rings R_1 and R_2 as follows.

Corollary 3.17. *Let X be a nodal curve; I a rank 1, torsion-free sheaf on X ; and Σ the set of nodes where I fails to be locally free. For every $e \in \Sigma$, fix an identification of $(\hat{\mathcal{O}}_{X,e}, I_e)$ with (\mathcal{O}_0, I_0) . Then the forgetful transformations in Eqn. (3.8) induce inclusions*

$$\widehat{\bigotimes_{e \in \Sigma} k[[U_e^{\leftarrow}, U_e^{\rightarrow}]] / (U_e^{\leftarrow} U_e^{\rightarrow})} \cong \widehat{\bigotimes_{e \in \Sigma} S_1} \hookrightarrow R_1,$$

$$\widehat{\bigotimes_{e \in \Sigma} k[[U_e^{\leftarrow}, U_e^{\rightarrow}, T_e]] / (U_e^{\leftarrow} U_e^{\rightarrow} - T_e)} \cong \widehat{\bigotimes_{e \in \Sigma} S_2} \hookrightarrow R_2,$$

and each inclusion realizes the larger ring as a power series ring over the smaller ring.

4. AUTOMORPHISM GROUPS AND THEIR ACTIONS

Automorphism groups appeared in the previous section, where we defined group actions on deformation functors (Def. 3.4). Here we study the structure of these groups with the aim of collecting results to use in §5. There we will study the problem of lifting the action of an automorphism group on a deformation functor to an action on a miniversal deformation ring. The existence of a lift follows from a theorem of Rim if the automorphism group is known to be linearly reductive. Thus, the focus of this section is on showing that the automorphism groups of interest are linearly reductive.

We begin by studying automorphisms of the node \mathcal{O}_0 (Def. 3.9) and its unique rank 1, torsion-free module I_0 that fails to be locally free (Def. 3.10). The automorphism group $\text{Aut}(X_0, I_0)$ fits into the exact sequence

$$(4.1) \quad 0 \longrightarrow \text{Aut}(I_0) \longrightarrow \text{Aut}(X_0, I_0) \longrightarrow \text{Aut}(X_0) \longrightarrow 0,$$

and the group $\text{Aut}(I_0)$ admits the following explicit description.

Lemma 4.1. *Consider I_0 as the normalization $\tilde{\mathcal{O}}_0$. Then the natural action of $\tilde{\mathcal{O}}_0^*$ on I_0 induces an isomorphism $\tilde{\mathcal{O}}_0^* \xrightarrow{\sim} \text{Aut}(I_0)$.*

Proof. We claim that every \mathcal{O}_0 -linear map $\phi: I_0 \rightarrow I_0$ is $\tilde{\mathcal{O}}_0$ -linear. It is enough to show that ϕ commutes with multiplication by $x/(x+y)$, and this is clear: for all $s \in I_0$, we have

$$(x+y) \cdot \phi(x/(x+y) \cdot s) = \phi(x \cdot s) = x \cdot \phi(s).$$

Dividing by $x+y$, we obtain the desired equality. Thus, $\text{Aut}(I_0)$ coincides with the group of $\tilde{\mathcal{O}}_0$ -linear automorphisms, which equals $\tilde{\mathcal{O}}_0^*$. \blacksquare

The action of $\tilde{\mathcal{O}}_0^*$ can also be described in terms of the presentation from Definition 3.10. A typical element $f \in \tilde{\mathcal{O}}_0^*$ can be uniquely written as $f = \alpha \frac{x}{x+y} + \beta \frac{y}{x+y} + g(x, y)$, with $\alpha, \beta \in k^*$ and $g(x, y) \in (x, y) \subset \mathcal{O}_0$, and this element acts by

$$e \mapsto (\alpha + g(x, 0))e,$$

$$f \mapsto (\beta + g(0, y))f.$$

We now turn to the global picture. Let I be a rank 1, torsion-free sheaf on a nodal curve X . Set Σ equal to the set of nodes where I fails to be locally free. In analogy with Eqn. (4.1), $\text{Aut}(X, I)$ fits into the following exact sequence:

$$(4.2) \quad 0 \longrightarrow \text{Aut}(I) \longrightarrow \text{Aut}(X, I) \longrightarrow \text{Aut}(X).$$

We describe $\text{Aut}(X, I)$ by describing the outermost groups.

Consider first $\text{Aut}(X)$. Without more information, we can only describe the rough features of this group. For X stable (the main case of interest), $\text{Aut}(X)$ is a finite, reduced group scheme ([DM69, Thm. 1.11]), and if we additionally assume that X is general and of genus $g \geq 3$, then this group is trivial. However, $\text{Aut}(X)$ can be highly non-trivial for special curves: see [vOV07] for a sharp bound on the cardinality of $\text{Aut}(X)$ in terms of the genus g , and for a description of the curves attaining the bounds.

The group $\text{Aut}(I)$ admits the following explicit description. In the notation from §3.2.1, there is a natural map $\text{Aut}(I) \rightarrow \text{Aut}(I_e)$ for every $e \in \Sigma$, and we use this map to describe $\text{Aut}(I)$.

Lemma 4.2. *Let X be a nodal curve; I a rank 1, torsion-free sheaf; Σ the set of points where I fails to be locally free; and $f: X_\Sigma \rightarrow X$ the map that resolves the nodes Σ . Then there is a unique isomorphism $H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}^*) \cong \text{Aut}(I)$ that extends the inclusion of $H^0(X, \mathcal{O}_X^*)$ in $\text{Aut}(I)$ and makes the diagram*

$$\begin{array}{ccc} H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}^*) & \xrightarrow{\cong} & \text{Aut}(I) \\ \downarrow & & \downarrow \\ \tilde{\mathcal{O}}_{X,e}^* & \xrightarrow{\cong} & \text{Aut}(I_e) \end{array}$$

commute for all $e \in \Sigma$. Here $\tilde{\mathcal{O}}_{X,e}$ is the normalization of the completed local ring at e , the horizontal maps are isomorphisms, and the vertical maps are restrictions.

Proof. Given I , we prove the stronger statement that $\underline{\text{End}}(I)$ is canonically isomorphic to $f_*(\mathcal{O}_{X_\Sigma})$. Because I is torsion-free, $\underline{\text{End}}(I)$ injects into $\underline{\text{End}}(I \otimes \text{Frac}(\mathcal{O}_X))$, which equals $\text{Frac}(\mathcal{O}_X)$ as I is rank 1. Thus, $\underline{\text{End}}(I)$ is a finitely generated, commutative \mathcal{O}_X -algebra satisfying $\mathcal{O}_X \subset \underline{\text{End}}(I) \subset \text{Frac}(\mathcal{O}_X)$. Furthermore, an application of the Cayley-Hamilton Theorem shows that a local section of $\underline{\text{End}}(I)$ satisfies a monic equation whose coefficients are local sections of \mathcal{O}_X . We may conclude that $\underline{\text{End}}(I) \subset g_*(\mathcal{O}_{\tilde{X}})$, where $g: \tilde{X} \rightarrow X$ is the (full) normalization. To complete the proof, it is enough to show that the support of $g_*(\mathcal{O}_{\tilde{X}})/\underline{\text{End}}(I)$ is precisely Σ . However, this can be checked on the level of completed stalks, and so we may deduce the claim from the Lemma 4.1. The result now follows by taking global sections of $\underline{\text{End}}(I)$ and passing to units. \blacksquare

One consequence of the previous two lemmas is that many of the groups appearing in this paper are linearly reductive. Recall that the ground field k may have positive characteristic, and in positive characteristic linear reductivity is a strong condition to impose. Indeed, while many algebraic groups (e.g. $\text{GL}_r, \text{SL}_r, \dots$) are linearly reductive in characteristic 0, Nagata has shown that the only linearly reductive groups in characteristic $p > 0$ are the groups G whose identity component G_0 is a multiplicative torus and whose étale quotient G/G_0 has prime-to- p order. We now list the groups we have shown satisfy this condition.

Definition 4.3. Let X be a nodal curve and I a rank 1, torsion-free sheaf generated by global sections. Define $\text{Aut}_1(I) \subset \text{Aut}(I)$ to be the subgroup of automorphisms $\tau \in \text{Aut}(I)$ with the property that the induced automorphism

$$H^0(X, I) \rightarrow H^0(X, I)$$

has determinant +1. Similarly, define $\text{Aut}_1(X, I) \subset \text{Aut}(X, I)$ to be the subgroup of automorphism $(\sigma, \tau) \in \text{Aut}(X, I)$ with the property that the induced automorphism

$$H^0(X, I) \xrightarrow{\text{can}} H^0(X, \sigma_*(I)) \xrightarrow{\tau} H^0(X, I)$$

has determinant +1.

Corollary 4.4. *Let X be a nodal curve and I a rank 1, torsion-free sheaf. Then the following groups are reduced and linearly reductive:*

- the automorphism group $\text{Aut}(I)$;
- the quotient group $\text{Aut}(I_0)/(1 + (x, y)\mathcal{O}_0)$;
- the automorphism group $\text{Aut}(X, I)$ when X is stable and does not admit an order p automorphism.

Assume further that I is generated by global sections. Then the same statement holds for the following groups:

- the group $\text{Aut}_1(I) \subset \text{Aut}(I)$;
- the group $\text{Aut}_1(X, I) \subset \text{Aut}(X, I)$ when X is stable and does not admit an order p automorphism.

Proof. Lemma 4.1 shows $\text{Aut}(I_0)/(1 + (x, y)\mathcal{O}_0)$ is a multiplicative torus, and Lemma 4.2 shows the same is true for $\text{Aut}(I)$. Given this, an inspection of Eqn. (4.2) proves that $\text{Aut}(X, I)$ is linearly reductive. We now consider the groups $\text{Aut}_1(I)$ and $\text{Aut}_1(X, I)$. If we label the connected components of X_Σ , then we can identify $\text{Aut}(I)$ with the standard torus \mathbb{G}_m^d and $\text{Aut}_1(I)$ with the subgroup scheme of \mathbb{G}_m^d consisting of sequences (g_1, \dots, g_d) satisfying $g_1 \dots g_d = 1$, which is a $(d - 1)$ -dimensional torus (parameterized by, say, $(g_1, \dots, g_{d-1}) \mapsto (g_1, \dots, g_{d-1}, (g_1 \dots g_{d-1})^{-1})$). Finally, $\text{Aut}_1(X, I)$ is an extension of $\text{Aut}_1(I)$ by a finite group of prime-to- p order. ■

5. GROUP ACTIONS ON RINGS

In this section we show that, in the cases of interest, the actions on deformation functors from Definition 3.4 lift to unique actions on miniversal deformation rings (Fact 5.4), which we then compute (Thm. 5.10). These results are used in §6, where we show that the action on the miniversal deformation ring can be described using the GIT construction of the compactified Jacobian (Lemma 6.4, Lemma 6.6). We then use this observation to deduce the main theorem of the paper (Thm. 6.1). Key to this section are the linear reductivity results from the previous section.

We begin by showing that certain actions are trivial.

Lemma 5.1. *The action of $1 + (x, y)\mathcal{O}_0 \subset \text{Aut}(I_0)$ on Def_{I_0} is trivial.*

Proof. Suppose we are given $A \in \text{Art}_k$ and a deformation (I_A, j) of I_0 over A . Given $\tau \in 1 + (x, y)\mathcal{O}_0$, we must show that (I_A, j) and $(I_A, \tau^{-1} \circ j)$ are isomorphic deformations. But this is clear: τ lies in \mathcal{O}_0 , and multiplication by $\tau \otimes 1 \in \mathcal{O}_0 \otimes_k A$ defines an isomorphism $(I_A, j) \xrightarrow{\sim} (I_A, \tau^{-1} \circ j)$. ■

Essentially the same argument proves the following two lemmas.

Lemma 5.2. *Let X be a nodal curve and I a rank 1, torsion-free sheaf on X . Then $\mathbb{G}_m \subset \text{Aut}(I)$ acts trivially on Def_I . Under the inclusion (4.2), \mathbb{G}_m also acts trivially on $\text{Def}_{(X, I)}$.*

Proof. We give a proof for Def_I ; the case of $\text{Def}_{(X, I)}$ is similar, and left to the reader. If (I_A, j) is a deformation of I and $\tau \in \mathbb{G}_m \subset \text{Aut}(I)$ a scalar automorphism, then τ trivially extends to an automorphism $\tilde{\tau}$ of I_A that defines an isomorphism of (I_A, j) with $(I_A, \tau^{-1} \circ j)$. ■

Lemma 5.3. *Let X be a nodal curve and I a rank 1, torsion-free sheaf. Then $\text{Aut}(I)$ acts trivially on the subfunctor $\text{Def}_I^{l.t.} \subset \text{Def}_I$. Under the inclusion (4.2), $\text{Aut}(I)$ also acts trivially on the subfunctor $\text{Def}_{(X, I)}^{l.t.} \subset \text{Def}_{(X, I)}$.*

Proof. The lemma is a consequence of Lemmas 4.1 and 3.16. ■

We may now invoke a theorem of Rim to show that the actions uniquely lift to actions on miniversal deformation rings.

Fact 5.4 (Rim [Rim80]). *Let X be a nodal curve and I a rank 1, torsion-free sheaf. Then:*

- (i) *there is a unique action of $\text{Aut}(I_0)$ on the miniversal deformation ring S_1 (resp. S_2) that makes the map $\text{Spf}(S_1) \rightarrow \text{Def}_{I_0}$ (resp. $\text{Spf}(S_2) \rightarrow \text{Def}_{(\mathcal{O}_0, I_0)}$) equivariant and has the property that the subgroup $1 + (x, y)\mathcal{O}_0 \subset \tilde{\mathcal{O}}_0^* = \text{Aut}(I_0)$ acts trivially;*
- (ii) *there is a unique action of $\text{Aut}(I)$ on the miniversal deformation ring R_1 of Def_I that makes $\text{Spf}(R_1) \rightarrow \text{Def}_I$ equivariant;*
- (iii) *there is a unique action of $\text{Aut}(X, I)$ on the miniversal deformation ring R_2 of $\text{Def}_{(X, I)}$ that makes $\text{Spf}(R_2) \rightarrow \text{Def}_{(X, I)}$ equivariant provided X is stable and does not admit an order p automorphism.*

Proof. The is a special case of [Rim80, pg. 225]. Indeed, the functors Def_{I_0} , Def_I , $\text{Def}_{(\mathcal{O}_0, I_0)}$ and $\text{Def}_{(X, I)}$ are all examples of a deformation functor F associated to a “homogeneous fibered category in groupoid” satisfying a finiteness condition. Given an action of a linearly reductive group on such a category, there is an induced action on F , and Rim’s Theorem asserts that there exists a miniversal deformation ring R that admits an action of G making $\text{Spf}(R) \rightarrow F$ equivariant. Furthermore, as an algebra with G -action, R is unique up to a (non-unique) isomorphism.

The proof is cohomological. By minimality R/\mathfrak{m}_R^2 admits a suitable action of G , and this action is inductively lifted to an action on R . Given a suitable action on R/\mathfrak{m}_R^n , the problem of lifting this action to an action on R/\mathfrak{m}_R^{n+1} is controlled by cohomology groups that are zero by linear reductivity.

One may verify that the actions on Def_{I_0} , Def_I , $\text{Def}_{(\mathcal{O}_0, I_0)}$ and $\text{Def}_{(X, I)}$ are defined on the level of groupoids. The claims concerning R_1 and R_2 follows immediately because we have shown that $\text{Aut}(I)$ and $\text{Aut}(X, I)$ are linearly reductive. The group $\text{Aut}(I_0)$ is certainly not linearly reductive, but Lemma 5.1 asserts that this group acts through its linearly reductive quotient $\text{Aut}(I_0)/(1 + (x, y)\mathcal{O}_0)$. Case (i) then follows as well. ■

The actions described by the Fact 5.4 are, of course, unique only up to a non-unique isomorphism. Because of the non-uniqueness, it is not immediate that the group action is functorial. This issue is addressed in the lemma below.

Lemma 5.5. *Let X be a nodal curve and I a rank 1, torsion-free sheaf. For every point $e \in X$ where I fails to be locally free, fix an isomorphism between $(\hat{\mathcal{O}}_{X, e}, I \otimes \hat{\mathcal{O}}_{X, e})$ and (\mathcal{O}_0, I_0) . Then the restriction transformations*

$$(5.1) \quad \begin{aligned} \text{Def}_I &\rightarrow \prod_{e \in \Sigma} \text{Def}_{I_0}, \\ \text{Def}_{(X, I)} &\rightarrow \prod_{e \in \Sigma} \text{Def}_{(\mathcal{O}_0, I_0)} \end{aligned}$$

lift to transformations of miniversal deformation rings

$$(5.2) \quad \begin{aligned} \text{Spf}(R_1) &\rightarrow \prod_{e \in \Sigma} \text{Spf}(S_1), \\ \text{Spf}(R_2) &\rightarrow \prod_{e \in \Sigma} \text{Spf}(S_2) \end{aligned}$$

that are equivariant with respect to the homomorphism

$$(5.3) \quad \text{Aut}(I) \rightarrow \prod_{e \in \Sigma} \text{Aut}(I_0)$$

and the actions of $\text{Aut}(I)$ and $\text{Aut}(I_0)$ described in Fact 5.4.

Proof. The only condition that is not immediate is that the natural transformations can be chosen to be equivariant. We give the proof for $\mathrm{Spf}(R_1)$ and leave the task of extending the argument to $\mathrm{Spf}(R_2)$ to the interested reader.

As $\mathrm{Spf}(S_1) \rightarrow \mathrm{Def}_{I_0}$ is formally smooth, there exists a lift $\mathrm{Spf}(R_1) \rightarrow \prod \mathrm{Spf}(S_1)$ of the forgetful transformation $\mathrm{Def}_I \rightarrow \prod \mathrm{Def}_{I_0}$, and such a lift is automatically formally smooth. Writing R_1 as a power series ring over $\widehat{\otimes} S_1$, it is easy to see that there exists an action of $\mathrm{Aut}(I)$ on $\mathrm{Spf}(R_1)$ that makes $\mathrm{Spf}(R_1) \rightarrow \prod \mathrm{Spf}(S_1)$ equivariant and has the property that the induced action on the tangent space $T_1(\mathrm{Spf}(R_1))$ coincides with the natural action on $T_1(\mathrm{Def}_I)$. To complete the proof, we must show that this action makes $\mathrm{Spf}(R_1) \rightarrow \mathrm{Def}_I$ equivariant, and hence satisfies the conditions of Fact 5.4.

Consider the composition $\mathrm{Spf}(R_1) \rightarrow \prod \mathrm{Def}_{S_1} \rightarrow \prod \mathrm{Def}_{I_0}$. This transformation is formally smooth and hence realizes R_1 as a (non-minimal) versal deformation ring for Def_{I_0} . Furthermore, the constructed action of $\mathrm{Aut}(I)$ on R_1 makes $\mathrm{Spf}(R_1) \rightarrow \prod \mathrm{Def}_{I_0}$ equivariant and induces the standard action on $T(R_1) = T(\mathrm{Def}_I)$. A second action on R_1 with this property is the unique action that makes $\mathrm{Spf}(R_1) \rightarrow \mathrm{Def}_I$ equivariant. An inspection of Rim's *proof* shows that the uniqueness statement in Fact 5.4 still holds if the miniversality hypothesis is weakened to versality, provided the action on the tangent space is specified. In particular, there is an automorphism of R_1 transforming the first action into the second. We can conclude that the map in (5.2) and the action in Fact 5.4 can be chosen so that $\mathrm{Spf}(S_1) \rightarrow \mathrm{Def}_I$ is equivariant. This completes the proof. ■

We now compute the actions described by Fact 5.4. Let us start with the action of $\mathrm{Aut}(I_0)$ on S_1 .

Lemma 5.6. *In terms of the presentation from Definitions 3.12, 3.11, define an action of $\mathrm{Aut}(I_0)$ on S_1 and S_2 by making $\tau = a\frac{x}{x+y} + b\frac{y}{x+y} + g \in \mathrm{Aut}(I_0)$ act as*

$$\begin{aligned} u &\mapsto ab^{-1} \cdot u, \\ v &\mapsto a^{-1}b \cdot v, \\ t &\mapsto t. \end{aligned}$$

Here $a, b \in k^*$ and $g \in (x, y)\tilde{\mathcal{O}}_0$. Then this action is the unique action described by Fact 5.4 (i).

Proof. We give a proof for the case of S_1 ; the case of S_2 is similar, and left to the reader. The rule above is easily seen to define an action of $\mathrm{Aut}(I_0)$ on S_1 with the property that $1 + (x, u)\mathcal{O}_0$ acts trivially, so we need only show that this action makes $\mathrm{Spf}(S_1) \rightarrow \mathrm{Def}_{I_0}$ into an equivariant map. In fact, it is enough to verify this for the subgroup of $\mathrm{Aut}(I_0)$ that consists of elements of the form $\tau := a\frac{x}{x+y} + b\frac{y}{x+y}$ because this subgroup maps isomorphically onto $\mathrm{Aut}(I_0)/(1 + (x, y)\mathcal{O}_0)$.

Given such a τ , what is the pullback of the miniversal deformation (I_{S_1}, i) under τ ? It is the module with presentation

$$(5.4) \quad \langle \tilde{e}', \tilde{f}' : y \cdot \tilde{e}' = -a^{-1}bu \cdot \tilde{f}', x \cdot \tilde{f}' = -ab^{-1}v \cdot \tilde{e}' \rangle,$$

together with the identification j sending $\tilde{e}' \mapsto e$, $\tilde{f}' \mapsto f$. One isomorphism between this deformation and the deformation $(I_{S_1}, \tau^{-1} \circ j)$ is

$$\begin{aligned} \tilde{e}' &\mapsto b^{-1}\tilde{e}, \\ \tilde{f}' &\mapsto a^{-1}\tilde{f}. \end{aligned}$$

This completes the proof. ■

We now turn our attention to the action of $\mathrm{Aut}(I)$ on R_1 . It is convenient to introduce some combinatorial language.

Definition 5.7. Let $e \in \Sigma$ be a node that lies on the intersection of the irreducible components v and w . Write \overleftarrow{e} for the pair (v, w) and \overrightarrow{e} for the pair (w, v) . Define $s, t: \{\overrightarrow{e}, \overleftarrow{e}\} \rightarrow \{v, w\}$ to be projection onto the first component and onto the second component respectively.

This notation is intended to be suggestive of graph theory. We may consider v and w as being vertices of the dual graph Γ_X that are connected by an edge corresponding to e . The pairs \overleftarrow{e} and \overrightarrow{e} should be thought of as orientations of this edge, and the maps s and t are the “source” and “target” maps sending an oriented edge to its source vertex and its target vertex respectively. The relation with graph theory is developed more systematically by the authors in [CMKV].

The group $\text{Aut}(I)$ can also be described using similar notation.

Definition 5.8. Let X be a nodal curve, I a rank 1, torsion-free sheaf, Σ the set of nodes where I fails to be locally free, and V the set of irreducible components of X . Define T_Σ to be the subgroup

$$T_\Sigma \subset \prod_{v \in V} \mathbb{G}_m$$

that consists of sequences (λ_v) with the property that $\lambda_{v_1} = \lambda_{v_2}$ for every two components v_1 and v_2 whose intersection contains some node *not* in Σ .

Remark 5.9. The torus T_Σ is isomorphic to $\text{Aut}(I) = H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}^*)$ (Lemma 4.2). Indeed, the element $\lambda = (\lambda_v) \in T_\Sigma$ corresponds to the regular function $f \in H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}^*)$ that is equal to the constant λ_v on the component v . It is convenient to have the following explicit isomorphism of $\text{Aut}(I)$ with a split torus. Let Γ_X be the dual graph of X and let $\Gamma = \Gamma_X(\Sigma)$ be the dual graph of a curve obtained from X by smoothing the nodes not in Σ . There is a morphism of graphs $c: \Gamma_X \rightarrow \Gamma$ ([CMKV, §2.1]) and it is easy to check there is an isomorphism

$$\phi: T_\Gamma := \prod_{v \in V(\Gamma)} \mathbb{G}_m \xrightarrow{\sim} T_\Sigma = \text{Aut}(I) \subseteq \prod_{w \in V(\Gamma_X)} \mathbb{G}_m$$

defined as follows. Given $(g_v) \in \prod_{v \in V(\Gamma)} \mathbb{G}_m$, set $\phi((g_v))_w = g_{c(w)}$ for each $w \in V(\Gamma_X)$.

We use the description of $\text{Aut}(I)$ as T_Σ to describe the action of $\text{Aut}(I)$ on R_1 and on R_2 .

Theorem 5.10. Let X be a nodal curve; I a rank 1, torsion-free sheaf; Σ the set of nodes where I fails to be locally free; and $g_\Sigma := h^1(X_\Sigma, \mathcal{O}_{X_\Sigma})$ the arithmetic genus of X_Σ . Then:

(i) Define an action of

$$T_\Sigma = \text{Aut}(I)$$

on

$$R_1(\Sigma) := k[[\{U_{\overleftarrow{e}}, U_{\overrightarrow{e}} : e \in \Sigma\}; W_1, \dots, W_{g_\Sigma}]] / (U_{\overleftarrow{e}} U_{\overrightarrow{e}} : e \in \Sigma)$$

by making $\lambda \in T_\Sigma$ act as

$$(5.5) \quad U_{\overrightarrow{e}} \mapsto \lambda_{s(\overrightarrow{e})} \cdot U_{\overrightarrow{e}} \cdot \lambda_{t(\overrightarrow{e})}^{-1},$$

$$(5.6) \quad U_{\overleftarrow{e}} \mapsto \lambda_{s(\overleftarrow{e})} \cdot U_{\overleftarrow{e}} \cdot \lambda_{t(\overleftarrow{e})}^{-1},$$

$$(5.7) \quad W_i \mapsto W_i.$$

Then there exists an isomorphism $R_1 \cong R_1(\Sigma)$ that identifies the above action of T_Σ on $R_1(\Sigma)$ with the action of $\text{Aut}(I)$ on R_1 from Fact 5.4.

(ii) Suppose $\text{Aut}(X)$ is trivial, and define an action of $T_\Sigma = \text{Aut}(X, I)$ on

$$R_2(\Sigma) := k[[\{U_{\overleftarrow{e}}, U_{\overrightarrow{e}}, T_e : e \in \Sigma\}; W_1, \dots, W_m]] / (U_{\overleftarrow{e}} U_{\overrightarrow{e}} - T_e : e \in \Sigma)$$

for some $m \in \mathbb{Z}_{\geq 0}$ by making $\lambda \in T_\Sigma$ act as in (5.5), (5.6), (5.7), and as $T_e \mapsto T_e$. Then there exists an isomorphism $R_2 \cong R_2(\Sigma)$ that identifies the above action of $T(\Sigma)$ on $R_2(\Sigma)$ with the action of $\text{Aut}(X, I)$ on R_2 from Fact 5.4.

Remark 5.11. Let $\Gamma = \Gamma_X(\Sigma)$ be the dual graph of any curve obtained from X by smoothing the nodes not in Σ . Then one can check that in the notation of the theorem above, $g_\Sigma = g(X) - b_1(\Gamma)$. It is also easy to see that the action of T_Γ on R_I and $R_{(X, I)}$ defined in Theorem A agrees with the action of T_Σ defined above.

Proof. This is a consequence of results already proven in this section. We only prove the statement about R_1 and leave the task of extending the proof to R_2 to the interested reader.

Suggestively set

$$(5.8) \quad S(\Sigma) := k[[U_{\leftarrow e}, U_{\rightarrow e} : e \in \Sigma]] / (U_{\leftarrow e} U_{\rightarrow e} : e \in \Sigma).$$

This is a miniversal deformation ring for $\prod \text{Def}_{I_0}$, where the product runs over the elements of Σ . If we fix an isomorphism between $(\widehat{\mathcal{O}}_{X, e}, I \otimes \widehat{\mathcal{O}}_{X, e})$ and (\mathcal{O}_0, I_0) for every node $e \in \Sigma$, then by Corollary 3.17 and Lemma 5.5, there exists an equivariant map $S(\Sigma) \hookrightarrow R_1$ realizing R_1 as a power series ring over $S(\Sigma)$. To complete the proof, we need to show that there exists an expression of R_1 as a power series ring generated by variables invariant under the group action.

Thus, consider the map from the cotangent space of $\text{Spf}(R_1)$ to the cotangent space of Def_I^{lt} . This is an equivariant map, and the action of $\text{Aut}(I)$ on the target space is trivial (Lemma 5.3). Because $\text{Aut}(I)$ is linearly reductive, we can find invariant elements $\bar{W}_1, \dots, \bar{W}_{g_\Sigma} \in R_1$ whose images in the cotangent space $\mathfrak{m}/\mathfrak{m}^2$ map isomorphically onto the cotangent space of Def_I^{lt} .

Letting W_1, \dots, W_{g_Σ} denote indeterminants, define a map

$$\phi: S(\Sigma)[[W_1, \dots, W_{g_\Sigma}]] \rightarrow R_1$$

by sending W_i to \bar{W}_i . The target and source of ϕ are isomorphic, and the induced map on tangent spaces is an isomorphism, hence ϕ itself must be an isomorphism.

Furthermore, if we make T_Σ act on $S(\Sigma)[[W_1, \dots, W_{g_\Sigma}]]$ by making the group act trivially on the indeterminants, then ϕ is equivariant. The ring $S(\Sigma)[[W_1, \dots, W_{g_\Sigma}]]$, together with this group action, is nothing other than $R_1(\Sigma)$, so the proof is complete. \blacksquare

Observe that the theorem computes the action of $\text{Aut}(X, I)$ on R_2 when X is automorphism-free. It would be interesting to compute the action when X is stable, but possibly admits non-trivial automorphisms. Indeed, such a result (combined with a suitable extension of Theorem 6.1) would allow us to remove the hypothesis that X does not have an automorphism from Theorem A. When X does not admit an automorphism of order p , Fact 5.4 states that there is a unique action of $\text{Aut}(X, I)$, so the problem is to modify the action described in Theorem 5.10 to incorporate $\text{Aut}(X)$.

The case where X admits an order p automorphism is more challenging for then we can no longer cite Rim's work to assert that $\text{Aut}(X, I)$ acts on R_2 or to assert that such an action, if it exists, is unique. Simply knowing if R_2 still admits a unique action of $\text{Aut}(X, I)$ would be interesting. More generally, it would be interesting to know if Rim's Theorem remains true if the assumption that the group G acting is linearly reductive is weakened.

6. LUNA SLICE ARGUMENT

We now prove that the invariant subrings in Theorem 5.10 are isomorphic to the completed local rings of the compactified Jacobians. The main result is the following.

Theorem 6.1. *Let X be a nodal curve and I a rank 1, torsion-free sheaf. Then:*

- (i) *Suppose $\bar{J}^d(X)$ is one of the following schemes:*

- a Caporaso–Pandharipande Jacobian of an automorphism-free stable curve;
- an Oda–Seshadri Jacobian;
- a Simpson Jacobian;

Assume I is poly-stable with respect to the associated stability condition. Then the action from Fact 5.4 of $\text{Aut}(I)$ on the deformation ring R_1 parameterizing deformations of I satisfies

$$\widehat{\mathcal{O}}_{\bar{J}_\phi(X),x} \cong R_1^{\text{Aut}(I)},$$

where $x \in \bar{J}^d(X)$ is the point corresponding to I .

- (ii) Assume X is stable and does not admit an order p automorphism, and I is slope poly-stable with respect to the dualizing sheaf ω_X . Then the action of $\text{Aut}(X, I)$ on the deformation ring R_2 satisfies

$$\widehat{\mathcal{O}}_{\bar{J}_{d,g},y} \cong R_2^{\text{Aut}(X,I)},$$

where $y \in \bar{J}^d(X)$ is the point of the universal compactified Jacobian corresponding to (X, I) .

In the theorem, the isomorphisms between the complete local rings are non-canonical, but this is necessarily so as the rings R_1 and R_2 are themselves only defined up to non-canonical isomorphism.

Remark 6.2. Observe that Theorem 6.1, together with Theorem 5.10, establishes Theorem A (see also Remarks 5.9, 5.11). An elementary argument in GIT shows that the ring $\widehat{B(\Gamma)}^{T_\Gamma}$ defined in Theorem A has dimension $b_1(\Gamma) + \#E(\Gamma)$. Since $\bar{J}_{d,g}$ has dimension $4g - 3$, it follows that $m = 4g - 3 - b_1(\Gamma) - \#E(\Gamma)$ in Theorem 5.10.

The proof of Theorem 6.1 is given at the end of the section, where it is deduced from the following sequence of lemmas.

Lemma 6.3. *Let X be a nodal curve; I a rank 1, torsion-free sheaf; and $q: \mathcal{O}_X^r \twoheadrightarrow I$ a surjection. If $H^1(X, I) = 0$, then the forgetful morphism $\text{Def}_q \rightarrow \text{Def}_I$ is formally smooth. Assume further that X is stable and $p: X \hookrightarrow \mathbb{P}^N$ is a 10-canonically embedded curve. Then $\text{Def}_{(p,q)} \rightarrow \text{Def}_{(X,I)}$ is formally smooth.*

Proof. We prove the statement about $\text{Def}_q \rightarrow \text{Def}_I$ and leave the proof for $\text{Def}_{(p,q)} \rightarrow \text{Def}_{(X,I)}$ to the interested reader. Given a surjection $B \twoheadrightarrow A$ of artin local k -algebras, a deformation (I_B, j) of I over B , and a deformation (q_A, j) of q such that the associated deformation of I is isomorphic to $(I_B \otimes_B A, j \otimes 1)$, we must show that there exists a deformation (q_B, j) extending (q_A, j) and inducing (I_B, j) . A filtering argument shows that the vanishing $H^1(X, I) = 0$ implies that $H^0(X_B, I_B) \rightarrow H^0(X_A, I_A)$ is surjective. Now suppose $s_1, \dots, s_r \in H^0(I_A)$ is the image of the standard basis for $H^0(X_A, \mathcal{O}_{X_A}^r)$. If we lift these elements to $\tilde{s}_1, \dots, \tilde{s}_r \in H^0(X_B, I_B)$ and define $q_B: \mathcal{O}_{X_B} \twoheadrightarrow I_B$ to be the map that sends the i -th standard basis element to \tilde{s}_i , then (q_B, j) has the desired properties. ■

We now relate R_1 and R_2 to the appropriate Quot schemes.

Lemma 6.4. *Let X be a nodal curve; I a rank 1, torsion-free sheaf; and $\tilde{x} \in \text{Quot}(\mathcal{O}_X^r)$ a point corresponding to a quotient map $q: \mathcal{O}_X^r \rightarrow I$. Assume:*

- $H^1(X, I) = 0$;
- $q: H^0(X, \mathcal{O}_X^r) \rightarrow H^0(X, I)$ is an isomorphism.

Then:

- (i) *If the orbit of \tilde{x} under the natural action of SL_r is closed in some invariant affine open subset $U \subseteq \text{Quot}(\mathcal{O}_X^r)$, the completed local ring $\widehat{\mathcal{O}}_{Z,\tilde{x}}$ of a Luna Slice Z of U through \tilde{x} is a miniversal deformation ring R_1 for Def_I .*

- (ii) Assume additionally that X is stable. Let $p: X \hookrightarrow \mathbb{P}^N$ be a 10-canonical embedding with (p, q) corresponding to the point \tilde{y} of the relative Quot scheme $\text{Quot}(\mathcal{O}_{X_g}^r)$ (from Sect. 2.8). If the orbit of \tilde{y} under the natural action of $\text{SL}_r \times \text{SL}_{N+1}$ is closed in some invariant affine open subset $V \subseteq \text{Quot}(\mathcal{O}_{X_g}^r)$, the completed local ring of $\widehat{\mathcal{O}}_{Z, \tilde{y}}$ of a Luna Slice Z of V through \tilde{y} is a miniversal deformation ring for $\text{Def}_{(X, I)}$.

Proof. We prove the statement relating $\text{Quot}(\mathcal{O}_X^r)$ to Def_I and leave the task of extending the argument to $\text{Def}_{(X, I)}$ to the interested reader. The necessary changes are primarily notational (e.g. the action of SL_r must be replaced with that of $\text{SL}_r \times \text{SL}_{N+1}$).

Temporarily set F equal to the functor pro-represented by $\widehat{\mathcal{O}}_{Z, x}$. There is a natural forgetful map $\text{Def}_q \rightarrow \text{Def}_I$, and our goal is to show that the restriction of this map to F is formally smooth and an isomorphism on tangent spaces. We do this by proving that $F(A) \rightarrow \text{Def}_I(A)$ is injective for $A = k[\epsilon]$ and has the same image as $\text{Def}_q(A) \rightarrow \text{Def}_I(A)$ for all $A \in \text{Art}_k$. Because $\text{Def}_q \rightarrow \text{Def}_I$ is formally smooth (Lemma 6.3), the lemma will then follow.

The desired facts are proven by studying the action of the lie algebra of SL_r on deformations. Set \mathfrak{sl}_r equal to the deformation functor pro-represented by the completed local ring of SL_r at the identity and \mathfrak{h} equal to the deformation functor associated to the stabilizer $H := \text{Stab}(x) \subset \text{SL}_r$. There is a natural map $\mathfrak{sl}_r/\mathfrak{h} \rightarrow \text{Def}_q$ given by the derivative of the orbit map. Concretely, this is defined by the rule $g \mapsto g \cdot v_{\text{triv}}$, where v_{triv} is the trivial deformation (over an unspecified artin local algebra). Because U admits a slice, there exists a morphism $\text{Def}_q \rightarrow \mathfrak{sl}_r/\mathfrak{h}$ that is a contraction onto the orbit in the sense that the derivative of the orbit map defines a section. Furthermore, this morphism has the property that the preimage of the trivial element $0 \in \mathfrak{sl}_r/\mathfrak{h}(A)$ is $F(A) \subset \text{Def}_q(A)$. The construction of the morphism is immediate: the scheme $Z \times_H \text{SL}_r$ admits a global contraction morphism given by projection onto the second factor, and the desired infinitesimal contraction is obtained by choosing a local inverse of $Z \times_H \text{SL}_r \rightarrow \text{Quot}(\mathcal{O}_X^r)$.

We can use the contraction morphism to deduce the second claim, that $\text{Def}_q(A) \rightarrow \text{Def}_I(A)$ and $F(A) \rightarrow \text{Def}_I(A)$ have the same image. Indeed, if $v \in \text{Def}_q(A)$ maps to an element of $\mathfrak{sl}_r/\mathfrak{h}(A)$ represented by $g \in \mathfrak{sl}_r(A)$, then $g^{-1} \cdot v$ lies in $F(A)$. Because both v and $g^{-1} \cdot v$ map to the same element of $\text{Def}_I(A)$, we have proven the claim.

We also need to verify that $F(k[\epsilon]) \rightarrow \text{Def}_I(k[\epsilon])$ is injective. This too can be proven using the contraction map, but we must first relate the kernel of $F(k[\epsilon]) \rightarrow \text{Def}_I(k[\epsilon])$ to the contraction. Specifically, we claim the kernel equals the image of the orbit map. It is immediate that the image is contained in the kernel, but the reverse inclusion requires more justification. Thus, suppose $(q_1: \mathcal{O}_{X_1}^r \rightarrow I_1, j)$ is a 1st order deformation with the property that (I_1, j) is the trivial deformation. Because q induces an isomorphism on global sections, we can choose bases and use the identification j to represent $q_1: H^0(\mathcal{O}_{X_1}^r) \rightarrow H^0(I_1)$ by a matrix g that reduces to the identity modulo ϵ . The matrix g may not lie in $\mathfrak{sl}_r(k[\epsilon])$, but if we set $\delta := \det(g)$, then the product $\delta \cdot g^{-1}$ does. One may check that $\delta \cdot g^{-1}$ maps to the deformation represented by $(q_1: \mathcal{O}_{X_1}^r \rightarrow I_1, j)$, establishing the reverse inclusion.

We now prove injectivity by showing directly that the image of the orbit map has trivial intersection with $F(k[\epsilon])$. Given v in this intersection, the image in $\mathfrak{sl}_r/\mathfrak{h}(k[\epsilon])$ under the contraction morphism is zero because v lies in $F(k[\epsilon])$. But, as v also lies in the image of the orbit map, the image under the composition $\text{Def}_q(k[\epsilon]) \rightarrow \mathfrak{sl}_r/\mathfrak{h}(k[\epsilon]) \rightarrow \text{Def}_q(k[\epsilon])$ of the contraction map with the orbit map is v . Thus, $v = 0$, and the proof is complete. ■

The following definition and lemma relate the stabilizer of a point of the Quot scheme to an automorphism group.

Definition 6.5. Let I be a rank 1, torsion-free sheaf; $q: \mathcal{O}_X^r \rightarrow I$ a quotient map corresponding to a point $\tilde{x} \in \text{Quot}(\mathcal{O}_X^r)$. Assume

- $q: H^0(X, \mathcal{O}_X^r) \rightarrow H^0(X, I)$ is an isomorphism.

Then:

- (i) If $\text{Stab}(\tilde{x}) \subset \text{SL}_r$ is the stabilizer under the natural action, then for $g \in \text{Stab}(\tilde{x})$, there is a unique automorphism $\alpha(g): I \rightarrow I$ with the property that

$$\alpha(g) \circ q = q \circ g^{-1}.$$

The **natural homomorphism**

$$\text{Stab}(\tilde{x}) \rightarrow \text{Aut}(I)$$

is defined by $g \mapsto \alpha(g)$. ($\alpha(g)$ exists by definition and is unique because I is generated by the image of $H^0(X, \mathcal{O}_X^r)$).

- (ii) Assume additionally that X is stable. Let $p: X \hookrightarrow \mathbb{P}^N$ be a 10-canonical embedding with (p, q) corresponding to the point \tilde{y} of the relative Quot scheme $\text{Quot}(\mathcal{O}_{X_g}^r)$ (from Sect. 2.8). If $\text{Stab}(\tilde{y}) \subset \text{SL}_r \times \text{SL}_{N+1}$ is the stabilizer under the natural action, then for $g = (g_1, g_2) \in \text{Stab}(\tilde{y})$, there is a unique automorphism $\alpha(g) = (\alpha_1(g), \alpha_2(g)) \in \text{Aut}(X, I)$ with the property that

$$\begin{aligned} p \circ \alpha_2(g) &= g_2 \circ p, \\ \alpha_1(g) \circ \alpha_2(g)_*(q) &= q \circ g_1^{-1}. \end{aligned}$$

The **natural homomorphism**

$$\text{Stab}(\tilde{y}) \rightarrow \text{Aut}(X, I)$$

is defined by $g \mapsto \alpha(g)$.

Lemma 6.6. *Let X be a nodal curve; I a rank 1, torsion-free sheaf; and $q: \mathcal{O}_X^r \rightarrow I$ a quotient corresponding to a point $\tilde{x} \in \text{Quot}(\mathcal{O}_X^r)$. Assume:*

- I is generated by global sections;
- $q: H^0(\mathcal{O}_X^r) \rightarrow H^0(I)$ is an isomorphism.

We have:

- (i) *The natural homomorphism*

$$\text{Stab}(\tilde{x}) \rightarrow \text{Aut}(I)$$

is injective with image $\text{Aut}_1(I)$.

- (ii) *Assume additionally that X is stable. Let $p: X \hookrightarrow \mathbb{P}^N$ be a 10-canonical embedding with (p, q) corresponding to the point \tilde{y} of the relative Quot scheme $\text{Quot}(\mathcal{O}_{X_g}^r)$ (from Sect. 2.8). Then the natural homomorphism*

$$\text{Stab}(\tilde{y}) \rightarrow \text{Aut}(X, I)$$

is injective with image $\text{Aut}_1(X, I)$.

Proof. As in the last proof, we only prove the statement for $\text{Stab}(\tilde{x})$ and leave the remaining details to the interested reader. Set $s_1, \dots, s_r \in H^0(X, I)$ equal to the image of the standard basis for $H^0(\mathcal{O}_X^r)$. We first show injectivity. Given $g \in \text{Stab}(\tilde{x})$, write $(a_{i,j}) := g^{-1}$. Then $\alpha(g)$ satisfies

$$(6.1) \quad \alpha(g)(s_i) = a_{i,1}s_1 + \dots + a_{i,r}s_r.$$

If $\alpha(g)$ is the identity, then we must have $\alpha(g)(s_i) = s_i$ for all i . But the s_i 's form a basis, so this is only possible if $g = \text{id}_r$, showing injectivity. Similarly, given an $\alpha \in \text{Aut}(I)$ that induces a determinant +1 automorphism of $H^0(I)$, define scalars $a_{i,j}$ as in Eqn. (6.1). Then $g := (a_{i,j})^{-1} \in \text{SL}_r$ is an element of $\text{Aut}_1(I)$ with $\alpha(g) = \alpha$. This completes the proof. \blacksquare

The last lemma we need asserts that the formation of the relevant group quotients commutes with completion.

Lemma 6.7. *Let Z be an affine algebraic scheme, $\tilde{x} \in Z$ a point, and H an algebraic group acting on Z that fixes \tilde{x} . Assume H is linearly reductive. Then the formation of H -invariants commutes with completion; i.e. we have*

$$\widehat{\mathcal{O}}_{Z,\tilde{x}}^H \cong \widehat{\mathcal{O}}_{Z/H,x}.$$

Here x is the image of \tilde{x} .

Proof. This is an exercise in linear reductivity. The quotient map induces a local homomorphism $\widehat{\mathcal{O}}_{Z/H,x} \rightarrow \widehat{\mathcal{O}}_{Z,\tilde{x}}$. Because \tilde{x} is a fixed point, H acts continuously on $\widehat{\mathcal{O}}_{Z,\tilde{x}}$, and passing to invariants, we may replace the target of this map with $\widehat{\mathcal{O}}_{Z,\tilde{x}}^H$. Our goal is to show that the resulting map is an isomorphism.

For injectivity, say $r \in \widehat{\mathcal{O}}_{Z/H,x}$ lies in the kernel. By picking a sequence $\{r_i\}_{i=1}^\infty$ in $\mathcal{O}_{Z/H,x}$ converging to r and studying the valuation of r_i , one can show that $r = 0$. Surjectivity requires more work.

Given $r \in \widehat{\mathcal{O}}_{Z,\tilde{x}}^H$, consider the reduction map $\widehat{\mathcal{O}}_{Z,\tilde{x}}^H \rightarrow \widehat{\mathcal{O}}_{Z,\tilde{x}}/\mathfrak{m}_{\tilde{x}}^{i+1}$. The element r maps to an H -invariant element \bar{r} in the target, which is canonically isomorphic to $\mathcal{O}_{Z,\tilde{x}}/\mathfrak{m}_{\tilde{x}}^{i+1}$. Fixing an equivariant splitting of $\mathcal{O}_{Z,\tilde{x}} \rightarrow \mathcal{O}_{Z,\tilde{x}}/\mathfrak{m}_{\tilde{x}}^{i+1}$ (which exists by linear reductivity), we can lift \bar{r} to an invariant element r_i of $\mathcal{O}_{Z,\tilde{x}}$. The collection of all these elements defines a sequence $\{r_i\}_{i=1}^\infty$ whose limit is r . Furthermore, every term in the sequence lies in $\widehat{\mathcal{O}}_{Z/H,x}$; thus the limit must lie in this ring as well. This completes the proof. \blacksquare

Proof of Theorem 6.1. The proof is an application of the Luna Slice Theorem, together with the previous lemmas. As usual, we only give the proof for a single compactified Jacobian and leave the task of extending the argument to the universal compactified Jacobian to the interested reader.

Let us begin by recalling the relevant details about the construction of the compactified Jacobian from §2.10. There we explained that we may assume the degree of I is sufficiently large and that the compactified Jacobian is the GIT quotient of an explicit open subscheme of $\text{Quot}(\mathcal{O}_X^r)$. The conditions on the subscheme imply that $H^1(I) = 0$ and there exists a quotient map $q: \mathcal{O}_X^r \rightarrow I$ inducing an isomorphism on global sections. This quotient map corresponds to a point \tilde{x} of the semi-stable locus $U \subset \text{Quot}(\mathcal{O}_X^r)$.

Because I is poly-stable, the orbit of \tilde{x} is closed, so a Luna Slice Z exists. Lemma 6.4 identifies the ring $\widehat{\mathcal{O}}_{Z,\tilde{x}}$ with the miniversal deformation ring R_1 . The image $\text{Aut}_1(I)$ of the natural homomorphism $\text{Stab}(\tilde{x}) \rightarrow \text{Aut}(I)$ naturally acts on R_1 . An exercise in unwinding definitions shows this action makes the structure map $\text{Spf}(R_1) \rightarrow \text{Def}_I$ equivariant. Thus, the action in question is the action described by Fact 5.4.

Lemma 6.7 allows us to relate the invariant subring to the completed local ring of the compactified Jacobian. Indeed, by definition the natural map from $Z/\text{Stab}(\tilde{x})$ to the compactified Jacobian induces an isomorphism on completed local rings, and the lemma identifies the completed local ring of $Z/\text{Stab}(\tilde{x})$ at \tilde{x} with $R_1^{\text{Aut}_1(I)}$.

We have now shown that the theorem holds, provided we replace the group $\text{Aut}(I)$ appearing with its subgroup $\text{Aut}_1(I)$. However, an inspection of, say, the description in Theorem 5.10 shows that the diagonal subgroup $\mathbb{G}_m \subset \text{Aut}(I)$ acts trivially on R_1 . Thus, the natural action of $\text{Aut}(I)$ on R_1 factors through the quotient $\text{Aut}(I)/\mathbb{G}_m$. Because the natural map $\text{Aut}_1(I) \rightarrow \text{Aut}(I)/\mathbb{G}_m$ is surjective, the $\text{Aut}_1(I)$ -invariants coincide with the $\text{Aut}(I)$ -invariants. This completes the proof. \blacksquare

In the introduction, we asked if Theorem 6.1 remains valid when X is allowed to have an automorphism of order p . The condition on the automorphism group was only used to apply the Luna Slice Theorem, which applies to actions of linearly reductive groups. It is probably unreasonable to expect an analogue of the Slice Theorem to hold for actions of an arbitrary reductive group (see

[MN01]), but we only need an analogue for actions of $\text{Aut}(X, I)$. This group is an extension of the finite (reduced) group $\text{Aut}(X)$ by the multiplicative torus $\text{Aut}(I)$, and it is known that the Slice Theorem holds for both the action of a torus (it is linearly reductive) and for the action of a finite group (e.g. [Gro63, Prop. 2.2]). Perhaps there is a Slice Theorem for actions of an extension of a torus by an arbitrary finite group?

Finally, we can prove Theorem B from the introduction.

Proof of Theorem B. Given Theorem A, this result follows from [CMKV]. To establish Parts (i) and (ii) of Theorem B, it is enough to fix a given poly-stable rank 1, torsion-free sheaf I corresponding a point x and prove the analogous statement about the completed local ring $\widehat{\mathcal{O}}_{\bar{J}^d(X), x}$. For the remainder of the proof, we will work exclusively with $\widehat{\mathcal{O}}_{\bar{J}^d(X), x}$.

Theorem A identifies $\widehat{\mathcal{O}}_{\bar{J}^d(X), x}$ with the invariant subring of the deformation ring R_1 . Let us recall the relation of that ring to the rings studied in [CMKV]. Say I fails to be locally free at a set of nodes Σ . If Γ is the dual graph of any curve obtained from X by smoothening the nodes not lying in Σ , then the edges of Γ are in natural bijection with the elements of Σ , and an inspection of the definitions in [CMKV] shows this correspondence induces identifications

$$\begin{aligned} R_1 &\cong \widehat{D}(\Gamma)[[W_1, \dots, W_{g_\Sigma}]], \\ T_\Sigma &\cong G_\Sigma \end{aligned}$$

respecting the relevant group actions. Here $\widehat{D}(\Gamma)$ is the completion of the ring $D(\Gamma)$ (from [CMKV, Sect. 7]) at the maximal ideal $\tilde{\mathfrak{m}} := (U_e^-, U_e^+ : e \in \Sigma)$.

By [CMKV, Thm. 7.1], the invariant subring of $D(\Gamma)$ is the cographic toric face ring $R(\Gamma)$ (from [CMKV, Def. 1.4]). Thus, we may cite Lemma 6.7 and Theorem A to assert that there is an isomorphism

$$\widehat{\mathcal{O}}_{\bar{J}^d(X), x} \cong \widehat{R}(\Gamma)[[W_1, \dots, W_{g_\Sigma}]]$$

of the completed local ring of $\bar{J}^d(X)$ with a power series ring over the completion of the cographic toric face ring $R(\Gamma)$ at the ideal $\mathfrak{m} := \tilde{\mathfrak{m}} \cap R(\Gamma)$ (which appears in [CMKV, Prop. 6.(i)]).

We now prove Part (i) of the theorem. In [CMKV], it is proven that $R(\Gamma)$ is Gorenstein ([CMKV, Thm. 8.1]) and has slc singularities ([CMKV, Thm. 8.7]), and these properties persists after passing to a completion and adjoining power series variables. We now turn our attention to Part (ii).

To establish (ii), it is enough to show that the multiplicity $e_{\mathfrak{m}}(R(\Gamma))$ equal 1 if and only if every element of Σ corresponds to a separating edge of the dual graph Γ_X of X . An inductive formula for $e_{\mathfrak{m}}(R(\Gamma))$ is given by [CMKV, Thm. 8.11] combined with [CMKV, Thm. 8.13], and the formula shows that $e_{\mathfrak{m}}(R(\Gamma)) = 1$ is equivalent to the condition that Γ is a tree. As Γ is obtained from Γ_X by contracting the edges not in Σ , the claim follows. This completes the proof. \blacksquare

7. EXAMPLES

In this section we present some examples to further elucidate the connections between the results in this paper, and those of [CMKV]. We will let X be a stable curve of genus g , and we will consider Simpson Jacobians $\bar{J} = \bar{J}_\omega^d(X)$ for various d . We refer the reader to Fact 2.6 for the connection between these compactified Jacobians and the others considered in this paper.

7.1. Integral curves. In the case of integral curves, the local structure of the compactified Jacobian is well understood to be a product of nodes and smooth factors. We review the case of a stable, integral curve with a single node using the results of this paper, as it provides a relatively simple example. The case with more nodes is similar.

Suppose X is an integral curve with a single node $p \in X$. Fix an integer d . Let I be a rank 1, torsion free sheaf on X of degree d corresponding to a point $x \in \bar{J}$. There are two possibilities,

either I is locally free, or it fails to be locally free exactly at p . In either case, one can check the sheaf is poly-stable. The dual graph Γ_X of X consists of a single node and a single edge. If I is locally free, then Theorem B immediately implies that \bar{J} is non-singular at x . On the other hand if I is not locally free, then in the notation of Theorem A, $\Sigma = \{p\}$ and $\Gamma = \Gamma_X(\Sigma) = \Gamma_X$. We see immediately that $A(\Gamma) = k[X, Y]/(XY)$. The action of T_Γ is trivial, and so the singularity of \bar{J} at x is étale locally a transverse intersection of smooth spaces.

From this description, we conclude that when I fails to be locally free, then étale locally at x there are two irreducible components of \bar{J} , the embedding dimension ($\dim T_x \bar{J}$) is $g + 1$, and the multiplicity ($\text{mult}_x \bar{J}$) is 2. We indicate how to recover these facts directly from the combinatorics of the graph Γ using the results of [CMKV]. First, using [CMKV, Corollary 6.3] and the fact that there are exactly two totally cyclic orientations on Γ (given by the two orientations of the graph; see [CMKV, Definition 2.3]), we recover that, étale locally, there are two irreducible components of \bar{J} at x . Next, using [CMKV, Theorem 8.10] and the fact that there are exactly two oriented circuits on Γ (given by the two orientations on the edge; see [CMKV, p.7]), we recover that the embedding dimension is $g + 1$. Finally, using [CMKV, Corollary 8.16], we see that $\text{mult}_x \bar{J} = 2$.

7.2. Two irreducible components. Let X_n be a curve with dual graph Γ_{X_n} consisting of two vertices and $n \geq 2$ edges connecting them. Such a graph, together with an (totally cyclic) orientation, is depicted below.

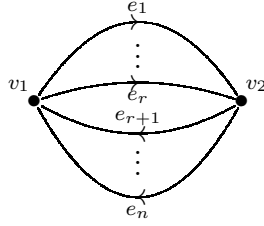


FIGURE 1. The orientation ϕ_r on Γ_{X_n} .

Fix an integer d , and suppose there exists a poly-stable sheaf I of degree d that fails to be locally free exactly at the n nodes of the curve; call them $p_1, \dots, p_n \in X_n$ and set $\Sigma = \{p_1, \dots, p_n\}$. Then $\Gamma_n := \Gamma_{X_n}(\Sigma) = \Gamma_{X_n}$. First we note that all of the totally cyclic orientations of Γ_n look like the orientation ϕ_r (for $1 \leq r \leq n - 1$) depicted in the figure above. In particular, there are

$$\sum_{r=1}^{n-1} \binom{n}{r}$$

totally cyclic orientations, so that \bar{J} has (étale locally) this many irreducible components. Second, there are $2\binom{n}{2}$ oriented circuits, so that the cographic ring $R(\Gamma_n)$ has this as its embedding dimension. Since \bar{J} has dimension g , and the dimension of the cographic ring $R(\Gamma_n)$ is equal to $b_1(\Gamma_n) = n - 1$, it follows that the embedding dimension of \bar{J} at x is

$$\dim T_x \bar{J} = 2\binom{n}{2} + (g - n + 1).$$

Finally, as explained in [CMKV, Example 8.17], it follows from [CMKV, Theorem 8.13, Corollary 8.16] that an (étale local) irreducible component of \bar{J} at x corresponding to a totally cyclic

orientation of Γ_n of type ϕ_r has multiplicity $\binom{n-2}{r-1}$ at x , and that

$$\text{mult}_x \bar{J} = \sum_{r=1}^{n-1} \binom{n}{r} \binom{n-2}{r-1}.$$

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